A TEXT-BOOK

ON THE

METHOD OF LEAST SQUARES.

BY

MANSFIELD MERRIMAN,

MEMBER OF AMERICAN MATHEMATICAL SOCIETY.

EIGHTH EDITION, REVISED.

TOTAL ISSUE, SEVEN THOUSAND.

NEW YORK

JOHN WILEY & SONS, INC.

LONDON: CHAPMAN & HALL, LIMITED

1913

THE SCIENTIFIC PRESS
ROBERT DRUMMOND AND COMPANY
BROOKLYN, N. Y.

PREFACE TO THE FIRST EDITION.

The "Elements of the Method of Least Squares," published in 1877, was written with two objects in view: first, to present the fundamental principles and processes of the subject in so plain a manner, and to illustrate their application by such simple and practical examples, as to render it accessible to civil engineers who have not had the benefit of extended mathematical training; and, secondly, to give an elementary exposition of the theory which would be adapted to the needs of a large and constantly increasing class of students.

In preparing the following pages the author has likewise kept these two objects continually in mind. While the former work has been used as a basis, the alterations and additions have been so numerous and radical as to render this a new and distinct book rather than a second edition. The arrangement of the theoretical and practical parts is entirely different. In Chapters I to IV is presented the mathematical development of the principles, methods, and formulas; while in Chapters V to IX the application of these

to the different classes of observations is made, and illustrated by numerous practical examples. For the use of both students and engineers, it is believed that this plan will prove more advantageous than the one previously followed. Hagen's deduction of the law of probability of error is given, as well as that of Gauss. More attention is paid to the laws of the propagation of error, the solution of normal equations, and the deduction of empirical formulas. Many new illustrative examples of the adjustment and comparison of observations have been selected from actual practice, and are discussed in detail. At the end of each chapter are given a few problems or queries; and in the Appendix are eight tables for abridging computations.

MANSFIELD MERRIMAN.

NOTE TO THE EIGHTH EDITION.

The seventh edition was the result of a thorough revision and was enlarged by the addition of new matter on the solution of normal equations, on the uncertainty of the probable error, and on the median. In this edition all known errors have been corrected and an alphabetical index has been added.

M. M.

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A TEXT-BOOK

ON THE

METHOD OF LEAST SQUARES.

CHAPTER I.

INTRODUCTION.

1. The Method of Least Squares has for its object the adjustment and comparison of observations. The adjustment of observations is rendered necessary by the fact, that when several precise measurements are made, even upon the same quantity under apparently similar conditions, the results do not agree. The absolutely true values of the observed quantities cannot in general be found, but instead must be accepted and used values, derived from the combination and adjustment of the measurements, which are the most probable, and hence the best. The comparison of observations is necessary in order to determine the relative degrees of precision of different sets of measurements made under different circumstances, either for the purpose of properly combining and adjusting them, or to ascertain the best methods of observation.

Classification of Observations.

2. Direct observations are those which are made directly upon the quantity whose magnitude is to be determined. Such are measurements of a line by direct chaining, or measurements of an angle by direct reading with a transit. They occur in the daily practice of every engineer.

Indirect observations are not made upon the quantity whose size is to be measured, but upon some other quantity or quantities related to it. Such are measurements of a line through a triangulation by means of a base and observed angles, measurements of an angle by regarding it as the sum or difference of other angles, the determination of the difference of level of points by readings upon graduated rods set up at different places, the determination of latitude by observing the altitude of stars, etc. In fact, the majority of observations in engineering and physical science generally belong to this class.

3. Conditioned observations may be either direct or indirect, but are subject to some rigorous requirement or condition imposed in advance from theoretical considerations. As such may be mentioned: the three measured angles in a plane triangle must be so adjusted that their sum shall be exactly 180°; the sum of all the percentages in a chemical analysis must equal 100; and the sum of the northings must equal the sum of the southings in any traverse which begins and ends at the same point.

Independent observations may be either direct or indirect, but are subject to no rigorous conditions. Measurements on two of the angles of a triangle, for instance, are independent; for the observed quantities can have no necessary geometrical dependence one upon the other.

4. As an illustration of these classes, consider the angles

AOB and *BOC*, having their vertices at the same point, *O* (Fig. 1). If a transit or theodolite be set at *O*, and the angle *AOB* or *BOC* be measured, each of these measurements is a direct observation. If, however, an auxiliary station *M* be established, and the angles *MOA*, *MOB*, and *MOC* be read, the observations on *AOB* and *BOC* are indirect. Moreover, whether observed directly or indirectly, the values obtained for *AOB* and *BOC* are independent of each other. But if the three angles *AOB*, *BOC*, and *AOC* be measured, these observations are conditioned, or subject to the rigorous geometrical requirement, that, when finally adjusted, *AOB* plus *BOC* must equal *AOC;* and no system of values can be adopted for these three angles which does not exactly satisfy this condition.

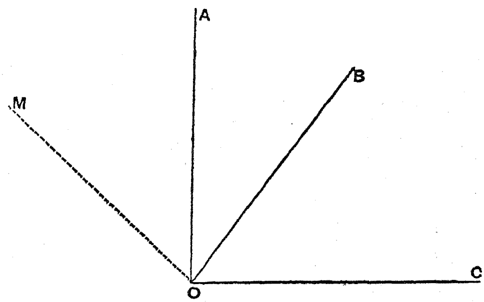

Fig. 1.

Again: if the sides and angles of a field are measured, each observation taken alone is direct. If its area is found from the sides and angles, the measurement of that area is indirect. Further: any two sides considered are independent of each other; but, if all the sides and angles be regarded, they must fulfil the condition, that, when plotted, they shall form a closed figure.

Errors of Observations.

5. Constant errors are those produced by well understood causes, and which may be removed from the observations by the application of computed corrections. As such may be

mentioned: theoretical corrections, like the effect of temperature upon the length of rods used in measuring a base-line; instrumental corrections, like those arising from a known discrepancy between the length of the rods and the standard of measure; and personal corrections, like those due to the habits of the observer, who, in making a contact of the rods, might err each time by the same constant quantity. Strictly speaking, then, constant errors are not errors; since they can always be eliminated from the observations, when the causes that produce them are understood. The first duty of an observer, after taking his measurements, is to discuss them, and apply as far as possible the computed corrections, to remove the constant errors.

6. Mistakes are errors committed by inexperienced and occasionally by the most skilled observers, arising from mental confusion. As such may be mentioned: mistakes in reading a compass-needle by noting 58° instead of 42°; or mistakes in measuring an angle by sighting at the wrong signal. Such errors often admit of correction by comparison with other sets of observations.

7. Accidental errors are those that still remain after all constant errors and all evident mistakes have been carefully investigated, and eliminated from the numerical results. Such, for example, are the errors in levelling arising from sudden expansions and contractions of the instrument, or from effects of the wind, or from the anomalous and changing refraction of the atmosphere. More than all, however, such errors arise from the imperfections of the touch and sight of the observer; which render it impossible for him to handle his instruments delicately, estimate accurately bisections of signals and small divisions of graduation, or keep them continually in adjustment. These are the errors that appear in all numerical observations, however carefully the measurements be made, and whose elimi-

nation is the object of the Method of Least Squares. Although at first sight it might seem that such irregular errors could not come within the province of mathematical investigation, it will be seen in the sequel that they are governed by a wonderful and very precise law, namely, the law of probability.

8. The word "error," as used in the following pages, means an accidental error produced by causes which are numerous, and whose effects cannot be brought within the scope of physical investigation. This error is the difference between the true value of the observed quantity and the result of the measurement upon it. Thus, if Z be the true value of an angle, and M_1, M_2, and M_3 be the results of measurements made upon it, the differences $Z - M_1$, $Z - M_2$, and $Z - M_3$ are the errors. An error is denoted by the letter x, and subscripts are applied to it for particular errors; thus, in the above case, $Z - M_1 = x_1$, $Z - M_2 = x_2$, and $Z - M_3 = x_3$, or, in general, x is the error of the observation M.

A residual is the difference between the most probable value of the observed quantity and the measurement upon it. This most probable value is that deduced by the application of the Method of Least Squares to the observations; for instance, in the simple case of direct measurements on a single quantity, the arithmetical mean is the most probable value. The residual is denoted in general by the letter v. Thus, if z be the most probable value of an angle derived from the measurements M_1, M_2, and M_3, the residuals are $z - M_1 = v_1$, $z - M_2 = v_2$, and $z - M_3 = v_3$. Evidently the most probable value, z, will approach more nearly to the true value Z, the greater the number of observations, as likewise the residuals v to the errors x. With an infinite number of precise observations, z should coincide with Z, and each v with the corresponding x. With a large number of observations, the differences between the resid-

uals and the errors will be small, so that the . aws governing the two will be essentially the same. On this account residuals are often called residual errors, or sometimes even errors.

Principles of Probability.

9. The word "probability," as used in mathematical reasoning, means a number less than unity, which is the ratio of the number of ways in which an event may happen or fail, to the total number of possible ways; each of the ways being supposed equally likely to occur. Thus, in throwing a coin, there are two possible cases: either head or tail may turn up, and one is as likely to occur as the other; hence the probability of throwing a head is expressed by the fraction $\frac{1}{2}$, and the probability of throwing a tail also by $\frac{1}{2}$. So, in throwing a die, there are six cases equally likely to occur, one of which may be the ace: hence the probability of throwing the ace in one trial is $\frac{1}{6}$, and the probability of not throwing it is $\frac{5}{6}$.

In general, if an event may happen in a ways, and fail in b ways, and each of these ways is equally likely to occur, the probability of its happening is $\frac{a}{a+b}$, and the probability of its failing is $\frac{b}{a+b}$. Thus probability is always expressed by an abstract fraction, and is a numerical measure of the degree of confidence which one has in the happening or failing of an event. As this measure may range from 0 to 1, so mental confidence may range from impossibility to certainty. If the fraction is 0, it denotes impossibility; if $\frac{1}{2}$, it denotes that the chances are equal for and against the happening of the event; and if 1, the event is certain to occur.

10. Unity is hence the mathematical symbol for certainty. And, since an event must either happen or not happen, the sum

of the probabilities of happening and failing is unity. Thus, if P be the probability that an event will happen, $1 - P$ is the probability of its failing. For example, if the probability of drawing a prize in a lottery is $\frac{1}{2000}$, the probability of not drawing a prize is $\frac{1999}{2000}$, a large fraction.

11. When an event may happen in different independent ways, the probability of its happening is the sum of the separate probabilities. For if it may happen in a ways, and also in a^{1} ways, and there are c total ways, the probability of its occurrence (by Art. 9) is $\dfrac{a + a^{1}}{c}$; and this is equal to the sum of the probabilities $\dfrac{a}{c}$ and $\dfrac{a^{1}}{c}$, of happening in the separate independent ways.

For example, if there be in a bag twenty red, sixteen white, and fourteen black balls, and one is to be drawn out, the probability that it will be red is $\frac{20}{50}$, that it will be white is $\frac{16}{50}$, and that it will be black is $\frac{14}{50}$. If, however, there be asked the probability of drawing either a red or black ball, the answer is $\frac{20}{50} + \frac{14}{50} = \frac{34}{50}$.

12. A compound event is one produced by the concurrence of several primary or simple events, each being independent of the other. For instance, throwing three aces with three dice in one trial is a compound event produced by the concurrence of three simple events. An error of observation may be regarded as a compound event produced by the combination of all the small independent errors of the numerous accidental influences.

The probability of the happening of a compound event is the product of the probabilities of the several primary independent events. To show this, consider two bags, one of which contains seven black and nine white balls, and the other four

black and eleven white balls. The probability of drawing a black ball from the first bag is $\frac{7}{16}$, and that of drawing one from the second $\frac{4}{15}$. What, now, is the probability of the compound event of securing two black balls when drawing from both bags at once? Since each ball in the first bag may form a pair with each one in the second, there are 16×15 possible ways of drawing two balls; and, since each of the seven black balls may be associated with each of the four black balls to form a pair, there are 7×4 cases favorable to drawing two black balls. The required probability is hence $\frac{7 \times 4}{16 \times 15}$; and this is equal to $\frac{7}{16} \times \frac{4}{15}$, or the product of the probabilities of the two primary independent events.

To discuss the principle more generally, consider two primary events, the first of which may happen in a_1 ways, and fail in b_1 ways, and the second happen in a_2, and fail in b_2 ways. Then there are for the first event $a_1 + b_1$ possible cases, and for the second $a_2 + b_2$; and each case out of the $a_1 + b_1$ cases may be associated with each case out of the $a_2 + b_2$ cases; and hence there are for the two events $(a_1 + b_1)(a_2 + b_2)$ total cases, each of which is equally likely to occur. In a_1a_2 of these cases both events happen; in b_1b_2 both fail; in a_1b_2 the first happens, and the second fails; and in a_2b_1 the first fails, and the second happens. Hence (by Art. 9) the probabilities of the compound events are—

Probability that both happen . . . . . . $\dfrac{a_1a_2}{(a_1+b_1)(a_2+b_2)}$

Probability that both fail . . . . $\dfrac{b_1b_2}{(a_1+b_1)(a_2+b_2)}$

Prob. that first happens, and second fails . $\dfrac{a_1b_2}{(a_1+b_1)(a_2+b_2)}$

Prob. that first fails, and second happens . $\dfrac{a_2b_1}{(a_1+b_1)(a_2+b_2)}$

As each of these probabilities is the product of the probabilities of the primary events, the principle is established for the case of two primary events. And evidently its extension to three or more is easy.

Thus, if there be four events, and P_1, P_2, P_3, and P_4 be the respective probabilities of happening, the probability that all the events will happen is $P_1 P_2 P_3 P_4$; and the probability that all will fail is $(1 - P_1)(1 - P_2)(1 - P_3)(1 - P_4)$. The probability that the first happens and the other three fail is $P_1(1 - P_2)(1 - P_3)(1 - P_4)$; and so on.

13. The most probable event among several is that which has the greatest mathematical probability. Thus, if two coins be thrown at the same time, there may arise the three following compound cases, having the respective probabilities as annexed:

Both may be heads	$\frac{1}{4}$
One head, and the other tail	$\frac{1}{2}$
Both tails	$\frac{1}{4}$

Here the case of one head and the other tail has the greatest probability, and is hence the most probable of the three compound events. The sum of the three probabilities, $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{1}{4}$, is unity; as should be the case, since one of these events is certain to occur.

If four measurements of the length of a line give the values 720.2, 720.3, 720.4, and 720.5 feet, the arithmetical mean, 720.35 feet, is universally recognized as the most probable value of the length of the line. It will be shown in the sequel that the mathematical probability of this result is greater than of any other.

14. A compound event, composed of any number of simple events, will now be considered. Let P be the probability of

the happening of an event in one trial, and Q the probability of its failing, so that $P+Q=1$: and let there be n such events. Then (by Art. 12) the probability that all will happen is P^n; the probability that one assigned event will fail, and $n-1$ happen. is $P^{n-1}Q$; and, since this may occur in n ways, the probability that one will fail, and $n-1$ happen, is $nP^{n-1}Q$. Similarly, the probability of two assigned events failing, and $n-2$ happening, is $P^{n-2}Q^2$; and, since this may be done in $\frac{n(n-1)}{2}$ ways,* the probability that two out of the whole number will fail, and $n-2$ happen, is $\frac{n(n-1)}{2}P^{n-2}Q^2$. If, then, $(P+Q)^n$ be expanded by the binomial formula, thus,

$$(P+Q)^n = P^n + nP^{n-1}Q + \frac{n(n-1)}{1.2}P^{n-2}Q^2 + \dots$$
$$+ \frac{n(n-1)(n-2)\dots(n-m+1)}{1.2.3\dots m}P^{n-m}Q^m + \text{etc.},$$

the first term is the probability that all will happen; the second, that $n-1$ will happen, and 1 fail; and the $m+1^{\text{th}}$ term is the probability that $n-m$ will happen, and m fail. To determine, then, the most probable case, it is only necessary to find the term in this series which is greatest.

The particular instance when $P=Q=\frac{1}{2}$ corresponds to the case of throwing n coins. Then the series becomes

$$(\tfrac{1}{2})^n + n(\tfrac{1}{2})^n + \frac{n(n-1)}{1.2}(\tfrac{1}{2})^n + \frac{n(n-1)(n-2)}{1.2.3}(\tfrac{1}{2})^n + \dots,$$

in which the middle term is the greatest if n be even, and

* See the theory of combinations in any algebra.

which has two equal middle terms if n be odd. Thus, if $n = 6$, the series is

$$\frac{1}{64} + \frac{6}{64} + \frac{15}{64} + \frac{20}{64} + \frac{15}{64} + \frac{6}{64} + \frac{1}{64}.$$

Hence, if six coins be thrown, the probabilities of the different cases are the following:

All heads	$\frac{1}{64}$
Five heads and one tail	$\frac{6}{64}$
Four heads and two tails	$\frac{15}{64}$
Three heads and three tails	$\frac{20}{64}$
Two heads and four tails	$\frac{15}{64}$
One head and five tails	$\frac{6}{64}$
All tails	$\frac{1}{64}$

The sum of these seven probabilities is, of course, unity.

15. The following graphical illustration gives a clear view of the relative values of the respective probabilities of the seven cases that may arise in throwing six coins. A horizontal straight line is divided into six equal parts, and at the points of division, ordinates are erected proportional to the probabilities $\frac{1}{64}$, $\frac{6}{64}$, etc., and through their extremities a curve is drawn. On the same diagram is shown, by a broken curve, the probabilities of the nine cases that may arise in throwing eight coins, or the terms

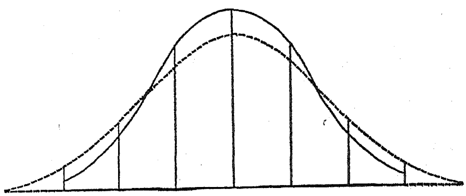

Fig. 2.

$$\frac{1}{256} + \frac{8}{256} + \frac{28}{256} + \frac{56}{256} + \frac{70}{256} + \text{, etc.,}$$

which are found by expanding the binomial $(\frac{1}{2} + \frac{1}{2})^8$.

It is one of the weaknesses of the human mind that large and small numbers do not convey to it accurate ideas unless aided by concrete analogy or representation. The above graphical illustration shows more clearly than the numbers themselves can do the relative probabilities in the two cases. These curves, moreover, are very similar to a curve hereafter to be discussed, which represents the law of probability of errors of observations.

Problems.

16. At the end of each chapter will be given a few questions and problems. The following will serve to exemplify the above principles of probability:

1. What is the probability of throwing an ace with a single die in two trials? Ans. $\frac{11}{36}$.

2. A bag contains three red, four white, and five black balls. Required the probability of drawing two red balls in two drawings, the ball first drawn not being replaced before the second trial?

3. Each student in a class of twenty is likely to solve one problem out of every eight. What is the probability that a given problem will be solved in the class?

4. What is the probability of throwing two aces, and no more, in a single throw with six dice? What is the probability of throwing at least two aces?

5. Let a hundred coins be thrown up each second by each of the inhabitants of earth. How often will a hundred heads be thrown in a million years?

6. A purse contains nine dimes and a nickel. A second purse contains ten dimes. Nine coins are taken from the first purse and put into the second, and then nine coins are taken from the second and put into the first. Which purse has the highest probable value?

CHAPTER II.

LAW OF PROBABILITY OF ERROR.

17. The probability of an assigned accidental error in a set of measurements is the ratio of the number of errors of that magnitude to the total number of errors. It is proposed, in this chapter, to investigate the relation between the magnitude of an error and its probability.

Axioms derived from Experience.

18. An analogy often referred to in the Method of Least Squares is that between bullet-marks on a target and errors of observations. The marksman answers to an observer; the position of a bullet-mark, to an observation; and its distance from the centre, to an error. If the marksman be skilled, and all constant errors, like the effect of gravitation, be eliminated in the sighting of the rifle, it is recognized that the deviations of the bullet-marks, or errors, are quite regular and symmetrical. First, it is observed that small errors are more frequent than large ones; secondly, that errors on one side are about as frequent as on the other; and, thirdly, that very large errors do not occur. Further: it is recognized, that, the greater the skill of the marksman, the nearer are the marks to his point of aim.

For instance, in the Report of the Chief of Ordnance for 1878, Appendix S′, Plate VI, is a record of one thousand shots fired deliberately (that is, with precision) from a battery-gun, at a target two hundred yards distant. The target was fifty-two

feet long by eleven feet high, and the point of aim was its central horizontal line. All of the shots struck the target; there being few, however, near the upper and lower edges, and nearly the same number above the central horizontal line as below it. On the record, horizontal lines are drawn, dividing the target into eleven equal divisions; and a count of the number of shots in each of these divisions gives the following results:

In top division	1 shot
In second division	4 shots
In third division	10 shots
In fourth division	89 shots
In fifth division	190 shots
In middle division	212 shots
In seventh division	204 shots
In eighth division	193 shots
In ninth division	79 shots
In tenth division	16 shots
In bottom division	2 shots
Total	1,000 shots

On Fig. 3 is shown, by means of ordinates, the distribution of these shots; A being the top division, B the middle, and C the bottom division. It will be observed that there is a slight preponderance of shots below the centre, and there is reason to believe that this is due to a constant error of gravitation not entirely eliminated in the sighting of the gun.

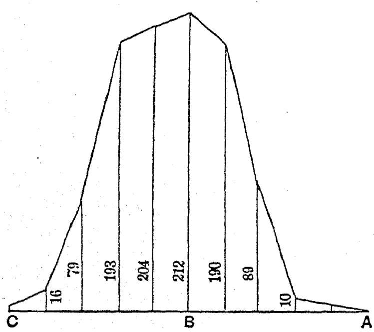

Fig. 3.

19. The distribution of the errors or residuals in the case of direct observations is similar to that of the deviations just

discussed. For instance, in the United States Coast Survey Report for 1854, p. *91, are given a hundred measurements of angles of the primary triangulation in Massachusetts. The residual errors (Art. 8) found by subtracting each measurement from the most probable values are distributed as follows:

Between $+6''.0$ and $+5''.0$	1 error
Between $+5.0$ and $+4.0$	2 errors
Between $+4.0$ and $+3.0$	2 errors
Between $+3.0$ and $+2.0$	3 errors
Between $+2.0$ and $+1.0$	13 errors
Between $+1.0$ and 0.0	26 errors
Between 0.0 and -1.0	26 errors
Between -1.0 and -2.0	17 errors
Between -2.0 and -3.0	8 errors
Between -3.0 and -4.0	2 errors
Total	100 errors

Here also it is recognized that small errors are more frequent than large ones, that positive and negative errors are nearly equal in number, and that very large errors do not occur. In this case the largest residual error was $5''.2$; but, with a less precise method of observation, the limits of error would evidently be wider.

20. The axioms derived from experience are, hence, the following:

> Small errors are more frequent than large ones.
> Positive and negative errors are equally frequent.
> Very large errors do not occur.

These axioms are the foundation of all the subsequent reasoning.

The Probability Curve.

21. In precise observations, then, the probability of a small error is greater than that of a large one, positive and negative

errors are equally probable, and the probability of a very large error is zero. The words "very large" may seem somewhat vague when used in general, although in any particular case the meaning is clear; thus, with a theodolite reading to seconds, $20''$ would be very large, and with a transit reading to minutes, $5'$ would be very large. Really, in every class of measurements there is a limit, l, such that all the positive errors are included between 0 and $+l$, and all the negative ones between 0 and $-l$.

22. Hence the probability of an error is a function of that error; so that, calling x any error and y its probability, the law of probability of error is represented by an equation

$$y = f(x),$$

and will be determined, if the form of $f(x)$ can be found. If, then, y be taken as an ordinate, and x as an abscissa, this may be regarded as the equation of a curve which must be of a form to agree with the three fundamental axioms; namely, its maximum ordinate OA must correspond to the error zero; it must be symmetrical with respect to the axis of Y, since positive and negative errors of equal magnitude are equally probable; as x increases numerically, the value of y must decrease, and, when x becomes very large, y must be zero. Fig. 4 represents such a curve, OP and OM being errors, and PB and MC their respective probabilities. Further: since different measurements have different degrees of accuracy, each class of observations will have a distinct curve of its own.

The curve represented in Fig. 4 is called the probability curve. In order to determine its equation, it is necessary to consider y as a continuous function of x. This is evidently perfectly allowable; since, as the precision of observations is increased, the successive values of x are separated by smaller and smaller intervals. The requirement of the third axiom, that y must be

zero for all values of x greater than the limit $\pm l$, is apparently an embarrassing one, as it is impossible to determine a continuous function of x which shall become zero for $x = \pm l$ and also be zero for all values of x from $\pm l$ to $\pm \infty$. But, since this limit l can never be accurately assigned, it will be best to extend the limits to $\pm \infty$, and determine the curve in such a way that

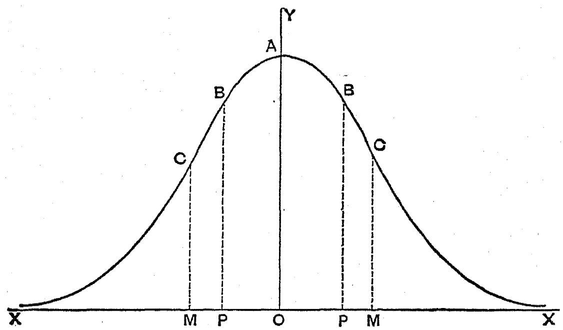

Fig. 4.

the value of y, although not zero for large values of x, will be so very small as to be practically inappreciable. The equation of the probability curve will be the mathematical expression of the law of probability of errors of observation. Two deductions of this law will be given; the first that of Hagen, and the second that of Gauss.

First Deduction of the Law of Error.

23. Hagen's demonstration rests on the following hypothesis or axiom, derived from experience:

An error is the algebraic sum of an indefinitely great number of small elementary errors which are all equal, and each of which is equally likely to be positive or negative.

To illustrate: suppose that, by several observations with a levelling instrument and rod, the difference in elevation between

two points has been determined. This value is greater or less than the true difference of level by a small error, x. This error x is the result of numerous causes acting at every observation: the instrument is not perfectly level, the wind shakes it, the sun's heat expands one side of it, the level-bubbles are not accurately made, the glass gives an indistinct definition, the tripod is not firm, the eye of the observer is not in perfect order, there is irregular refraction of the atmosphere, the man at the rod does not hold it vertical, the turning-points are not always good ones, the graduation of the rod is poor, the target is not properly clamped, the rod-man errs in taking the reading, and many others. Again: each of these causes may be subdivided into others; for instance, the error in reading the rod may be due, perhaps, to the accumulated result of hundreds of little causes. The total error, x, may hence be fairly regarded as resulting from the combination of an indefinitely great number of small elementary errors; and no reason can be assigned why one of these should be more likely to be positive than negative, or negative than positive.

24. Now, it is evident that it is more probable that the number of positive elementary errors should be approximately equal to the number of negative ones than that either should be markedly in excess, and that the probability of the elementary errors being either all positive or all negative is exceedingly small. In the first case the actual error is small, and in the second large; and so the probabilities of small errors are the greatest, and the probability of a very large error is practically zero. These correspond to the properties which the probability curve must possess.

Let Δx represent the magnitude of an elementary error, and m the number of those errors. The probability that any Δx will be positive is $\frac{1}{2}$, and that it will be negative is also $\frac{1}{2}$. The probability that all of the m elementary errors will be positive

is hence $(\frac{1}{2})^m$; the probability that $m - 1$ will be positive and 1 negative is $m(\frac{1}{2})^{m-1}(\frac{1}{2})^1$; and the probabilities of all the respective cases will be given by the corresponding terms of the binomial formula (Art. 14). When all of the m elementary errors are positive, the resulting error of observation is $+ m.\Delta x$; when $m - 1$ are positive and 1 negative, the resulting error is $+ (m - 1)\Delta x - \Delta x$, or $+ (m - 2)\Delta x$. If $m - n$ elementary errors are positive and the remaining n are negative, the resulting error is $+ (m - n)\Delta x - n.\Delta x$, or $+ (m - 2n)\Delta x$, and the probability of this particular combination is given by the $n + 1^{\text{th}}$ term of the expansion of the binomial $(\frac{1}{2} + \frac{1}{2})^m$. It is easy then to write the following table:

Elementary Errors Δx.	Resulting Error x.	Its Probability y.
If m are + and 0 are −	$m\Delta x$	$\left(\frac{1}{2}\right)^m$
If $m-1$ are + and 1 is −	$(m-2)\Delta x$	$m\left(\frac{1}{2}\right)^m$
If $m-2$ are + and 2 are −	$(m-4)\Delta x$	$\frac{m(m-1)}{1.2}\left(\frac{1}{2}\right)^m$
If $m-3$ are + and 3 are −	$(m-6)\Delta x$	$\frac{m(m-1)(m-2)}{1.2.3}\left(\frac{1}{2}\right)^m$
. . . .	. . . . . .	. . . .
If $m-n$ are + and n are −	$(m-2n)\Delta x$	$\frac{m(m-1)(m-2)\ldots(m-n+1)}{1.2.3\ldots n}\left(\frac{1}{2}\right)^m$
If $m-n-1$ are + and $n+1$ are −	$(m-2n-2)\Delta x$	$\frac{m(m-1)(m-2)\ldots(m-n)}{1.2.3\ldots n+1}\left(\frac{1}{2}\right)^m$
. . . .	. . . . . .	. . . .

25. In the curve $y = f(x)$ let OM be any error x, and MC its probability y; also let OP be an error x' less in magnitude, and PB its corresponding probability y'. Then, from the figure,

$$\text{limit}\ \frac{BD}{CD} = \text{limit}\ \frac{y - y'}{x - x'} = \frac{dy}{dx}$$

is the differential equation of the curve. To deduce, then, the law of probability of error, it is only necessary to find $\frac{y-y'}{x-x'}$ in terms of y and x, pass to the limit, place it equal to $\frac{dy}{dx}$, and perform the integration.

If x' be taken as the error next less in magnitude to x, the difference $x-x'$ equals $2\Delta x$, and the value of $\frac{y-y'}{x-x'}$ is the limit $\frac{dy}{dx}$ if the curve is to be continuous.

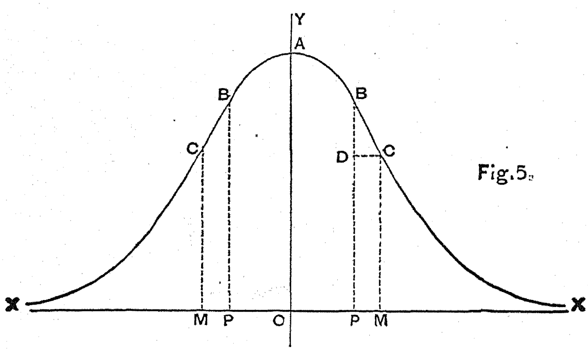

Fig. 5.

26. For the two consecutive errors x and x' take (from Art. 24) the two general values

$$x = (m - 2n)\Delta x, \quad \text{and} \quad x' = (m - 2n - 2)\Delta x.$$

The ratio of the probabilities of these errors is

$$\frac{y'}{y} = \frac{m-n}{n+1},$$

which, after inserting for n its value in terms of x, m, and Δx, may be put into the form

$$y - y' = y\frac{2(\Delta x - x)}{(m+2)\Delta x - x} = y\frac{-2x}{m\Delta x}.$$

Here Δx in the numerator vanishes in comparison with x. In the denominator, 2 vanishes compared with m, and $m\Delta x$ is the maximum positive error, which is so large that x vanishes in comparison with it. The differential equation, then, is

$$\frac{dy}{dx} = \frac{y - y'}{2\Delta x} = -\frac{yx}{m\Delta x^2},$$

or

$$\frac{dy}{dx} = -2h^2yx,$$

in which $2h^2$ has been written to represent the quantity $\frac{1}{m\Delta x^2}$. The integration of this equation gives

$$\log y = -h^2x^2 + k',$$

in which k' is the constant of integration, and the logarithm is in the Napierian system. By passing from logarithms to numbers

$$y = e^{-h^2x^2 + k'} = e^{-h^2x^2}e^{k'},$$

in which e is the base of the Napierian system. Since $e^{k'}$ is a constant, this may be written

$$(1) \qquad y = ke^{-h^2x^2},$$

and this is the equation of the probability curve, or the equation expressing the law of probability of errors of observation.

This equation satisfies the conditions imposed in Art. 22, for y is a maximum when x is 0; it is symmetrical with respect to the axis of Y, since equal positive and negative values of x give equal values of y, and when x becomes very large, y is very small. The constants k and h will be particularly considered hereafter.

Second Deduction of the Law of Error.

27. Gauss's demonstration is based on the following hypothesis or axiom, established by experience:

The most probable value of a quantity which is observed directly several times, with equal care, is the arithmetical mean of the measurements.

The average or arithmetical mean has always been accepted and used as the best rule for combining direct observations of equal precision upon one and the same quantity. This universal acceptance may be regarded as sufficient to justify the axiom that it gives the most probable value, the words "most probable" being used in the sense of Art. 13; for after all, as Laplace has said, the theory of probability is nothing but common sense reduced to calculation. If the measurements be but two in number, the arithmetical mean is undoubtedly the most probable value; and, for a greater number, mankind, from the remotest antiquity, has been accustomed to regard it as such.

It is a characteristic of the arithmetical mean that it renders the algebraic sum of the residual errors zero. To show this, let M_1, $M_2 \dots M_n$, be n measurements of a quantity; then the arithmetical mean of these is,

$$z = \frac{M_1 + M_2 + M_3 + \dots + M_n}{n}.$$

This equation may be written

$$nz = M_1 + M_2 + M_3 + \dots + M_n,$$

which by transposition becomes

$$(z - M_1) + (z - M_2) + (z - M_3) + \dots + (z - M_n) = 0;$$

that is to say, the arithmetical mean requires that the algebraic

sum of the residual errors shall be zero. To take a numerical illustration, let 730.4, 730.5, and 730.9 be three measurements of the length of a line. The arithmetical mean is 730.6, giving the residuals $+0.2$, $+0.1$, and -0.3, whose algebraic sum is 0.

28. Consider the general case of indirect observations, in which it is required to find the most probable values of quantities by measurements on functions of those quantities. For simplicity, only two quantities, z_1 and z_2, will be considered; although the reasoning is general, and applies to any number. Let n observations be made on functions of z_1 and z_2, from which it is required to find the most probable values of z_1 and z_2. The differences between the observations and the corresponding true values of the functions are errors $x_1, x_2 \ldots x_n$, each of which is also a function of z_1 and z_2. The probabilities of these errors are

$$y_1 = f(x_1), \quad y_2 = f(x_2) \ldots y_n = f(x_n).$$

And by Art. 12 the probability of committing the given system of errors is

$$P = y_1 y_2 y_3 \ldots y_n = f(x_1)\, f(x_2) \ldots f(x_n).$$

Applying logarithms to this expression, it becomes

$$\log P = \log f(x_1) + \log f(x_2) + \ldots + \log f(x_n).$$

Now, the most probable values of the unknown quantities z_1 and z_2 are those which render P a maximum (Art. 13), and hence the derivative of P with respect to each of these variables must be equal to zero. Indicating the differentiation, the following equations result:

$$\frac{dP}{Pdz_1} = \frac{df(x_1)}{f(x_1)\,dz_1} + \frac{df(x_2)}{f(x_2)\,dz_1} + \ldots + \frac{df(x_n)}{f(x_n)\,dz_1} = 0,$$

$$\frac{dP}{Pdz_2} = \frac{df(x_1)}{f(x_1)\,dz_2} + \frac{df(x_2)}{f(x_2)\,dz_2} + \ldots + \frac{df(x_n)}{f(x_n)\,dz_2} = 0.$$

Since in general $df(x) = \phi(x)f(x)dx$, these may be written

$$\phi(x_1)\frac{dx_1}{dz_1} + \phi(x_2)\frac{dx_2}{dz_1} + \dots + \phi(x_n)\frac{dx_n}{dz_1} = 0,$$

$$\phi(x_1)\frac{dx_1}{dz_2} + \phi(x_2)\frac{dx_2}{dz_2} + \dots + \phi(x_n)\frac{dx_n}{dz_2} = 0,$$

and, being as many in number as there are unknown quantities, they will determine the values of those unknown quantities as soon as the form of the function ϕ is known.

Since these equations are general, and applicable to any number of unknown quantities, the form of the function ϕ may be determined from any special but known case. Such is that in which there is but one unknown quantity, and the observations are taken directly upon that quantity. Thus, if there be only the quantity z, and the measurements give for it the values M_1, $M_2 \dots M_n$, the errors are,

$$x_1 = z - M_1,\ x_2 = z - M_2 \dots x_n = z - M_n,$$

from which

$$\frac{dx_1}{dz} = \frac{dx_2}{dz} = \dots = \frac{dx_n}{dz} = 1,$$

and the first equation above becomes

$$\phi(x_1) + \phi(x_2) + \phi(x_3) + \dots + \phi(x_n) = 0.$$

In this case, also, the arithmetical mean is the most probable value, and the algebraic sum of the residuals will be zero, or, if v denote any residual in general,

$$v_1 + v_2 + v_3 + \dots + v_n = 0.$$

Now, if the number of observations, n, is very large, the residuals v will coincide with the errors x (Art. 8), and

$$x_1 + x_2 + x_3 + \dots + x_n = 0.$$

This equation can only agree with that above when ϕ signifies multiplication by a constant, or when

$$\phi(x_1) + \phi(x_2) + \ldots + \phi(x_n) = cx_1 + cx_2 + \ldots + cx_n.$$

Replacing in this the values of $\phi(x_1)$, $\phi(x_2)$, etc., it becomes

$$\frac{df(x_1)}{f(x_1)dx_1} + \frac{df(x_2)}{f(x_2)dx_2} + \text{etc.} = cx_1 + cx_2 + \text{etc.};$$

and, since this is true whatever be the number of observations, the corresponding terms in the two members are equal. Hence, if x be any error, and $y = f(x)$,

$$\frac{df(x)}{f(x)dx} = \frac{dy}{ydx} = cx.$$

Multiplying both members by dx, and integrating,

$$\log y = \frac{cx^2}{2} + k',$$

Passing from logarithms to numbers,

$$y = e^{\frac{1}{2}cx^2} e^{k'}.$$

Here the constant c must be essentially negative, since the probability y should decrease as x increases numerically; replacing it, then, by $-2h^2$, and also putting $e^{k'} = k$, there results

$$(1) \qquad y = ke^{-h^2x^2},$$

which is the equation of the probability curve, or the equation expressing the law of probability of errors of observation.

Discussion of the Curve $y = ke^{-h^2x^2}$.

29. Since positive and negative values of x numerically equal give equal values of y, the curve is symmetrical with respect to the axis of Y. The maximum value of y is for $x = 0$, when

$y = k$; k is, hence, the probability of the error 0. As x increases numerically, y decreases; and when $x = \infty$, y becomes 0. The value of the first derivative is

$$\frac{dy}{dx} = -2kh^2e^{-h^2x^2}x,$$

which becomes zero when $x = 0$ and when $x = \pm\infty$, indicating that the curve is horizontal over the origin, and that the axis of x is an asymptote. The value of the second derivative is

$$\frac{d^2y}{dx^2} = -2kh^2e^{-h^2x^2}(-2h^2x^2 + 1),$$

which becomes 0 when $-2h^2x^2 + 1 = 0$, indicating that the curve has an inflection-point when $x = \pm\frac{1}{h\sqrt{2}}$.

To show further the form of the curve, the following values have been computed, taking k and h each as unity:

$$y = e^{-x^2} = \frac{1}{e^{x^2}}.$$

x	y	x	y
.0	1.0000	±1.8	0.0392
±0.2	0.9608	±2.0	0.0183
±0.4	0.8521	±2.2	0.0079
±0.6	0.6977	±2.4	0.0032
±0.8	0.5273	±2.6	0.0012
±1.0	0.3679	±2.8	0.0004
±1.2	0.2370	±3.0	0.0001
±1.4	0.1409		
±1.6	0.0773	±∞	0.0000

The curve in Fig. 4 is constructed from these values, the vertical scale being double the horizontal. C is the inflection-point, whose abscissa OM is 0.707.

30. The constant h is a quantity of the same kind as $\frac{1}{x}$, since the exponent h^2x^2 must be an abstract number. Methods will be hereafter explained by which its value may be determined for given observations. The probability of an assigned error x' decreases as h increases; and hence, the more precise the observations, the greater is h. For this reason h may be called "the measure of precision."

The constant k is an abstract number; and, since it is the probability of the error 0, it is larger for good observations than for poor ones. The more precise the measurements, the larger is k.

The Probability Integral.

31. To determine the value of the constant k, and also to investigate the probability of an error falling between assigned limits, the following reasoning may be employed:

Let $x', x_1, x_2 \ldots x$ be a series of errors, x' being the smallest, x_1 the next following, and x the last; the differences between the successive values being equal, and x' being any error. Then, by Art. 11, the probability of committing one of these errors, that is, the probability of committing an error lying between x' and x, is the sum of the separate probabilities $ke^{-h^2x'^2}$, $ke^{-h^2x_1^2}$, etc.; or, if P denote this sum,

$$P = k(e^{-h^2x'^2} + e^{-h^2x_1^2} + e^{-h^2x_2^2} + \ldots + e^{-h^2x^2}),$$

which may be written

$$P = k\Sigma_{x'}^{x} e^{-h^2x^2},$$

the notation $\Sigma_{x'}^{x}$ denoting summation from x' to x inclusive.

To replace the sign of summation by that of integration, dx must be the interval between the successive values of the errors, and then the probability that an error will lie between any two limits x' and x is

$$P = \frac{k}{dx}\int_{x'}^{x} e^{-h^2x^2}dx.$$

Now, it is certain that the error will lie between $-\infty$ and $+\infty$, and, as unity is the symbol for certainty,

$$1 = \frac{k}{dx}\int_{-\infty}^{+\infty} e^{-h^2x^2}dx.$$

The value of the definite integral in this expression is $\frac{\sqrt{\pi}}{h}$.*

Hence

$$1 = \frac{k\sqrt{\pi}}{hdx},$$

---

* The following method of determining this integral is nearly that presented by Sturm in his Cours d'Analyse, Paris, 1857, vol. ii. p. 16.

The integral $\int e^{-h^2x^2}dx$ expresses the area between the probability curve and the axis of X; and, since the curve is symmetrical to the axis of Y, that integral between the limits $-\infty$ and $+\infty$ will be equal to double the integral between the limits 0 and $+\infty$. Placing also $hx = t$,

$$\int_{-\infty}^{+\infty} e^{-h^2x^2}dx = \frac{2}{h}\int_0^{\infty} e^{-t^2}dt,$$

and the integral in the second member is to be determined.

Take three co-ordinate rectangular axes OT, OU, and OV, and change t into u, then

$$A = \int_0^{\infty} e^{-t^2}dt = \text{area between curve } VtT \text{ and axes,}$$

$$A = \int_0^{\infty} e^{-u^2}du = \text{area between curve } VuU \text{ and axes,}$$

and
$$A^2 = \int_0^{\infty}\int_0^{\infty} e^{-t^2-u^2}dtdu.$$

from which the value of k is

$$k = \frac{hdx}{\sqrt{\pi}}.$$

The equation of the probability curve now becomes

$$(2) \qquad y = hdx\pi^{-\frac{1}{2}}e^{-h^2x^2},$$

and the probability that an error lies between any two given limits x' and x becomes

$$(3) \qquad P = \frac{h}{\sqrt{\pi}}\int_{x'}^{x} e^{-h^2x^2}dx.$$

Equations (1), (2), and (3) are the fundamental ones in the theory of accidental errors of observation.

32. The probability that an error lies between the limits $-x$ and $+x$ is double the probability that it lies between the limits 0 and $+x$, on account of the symmetry of the curve. Hence

$$(4) \qquad P = \frac{2h}{\sqrt{\pi}}\int_{0}^{x} e^{-h^2x^2}dx$$

Now $v = e^{-t^2}$ is the equation of the curve VtT, and $v = e^{-u^2}$ is the equation of VuU, and, if either of these curves revolves about the axis of V, it generates a surface whose equation is $v = e^{-t^2-u^2}$. Hence the double integral A^2 is one-fourth of the volume included between that surface and the horizontal plane. If a series of cylinders concentric with the axis V form the volume, the area of the ring included between two whose radii are r and $r + dr$ is $2\pi rdr$, and the corresponding height is $v = e^{-t^2-u^2} = e^{-r^2}$. Hence one-fourth of the volume is

$$A^2 = \frac{1}{4}\int_0^\infty e^{-r^2}2\pi rdr,$$

which, since $\int e^{-r^2}2rdr = e^{-r^2}$, is equal to $\frac{\pi}{4}$. Therefore

$$A = \int_0^\infty e^{-t^2}dt = \frac{\sqrt{\pi}}{2},$$

and hence, finally,

$$\int_{-\infty}^{+\infty} e^{-h^2x^2}dx = \frac{2}{h}\int_0^\infty e^{-t^2}dt = \frac{\sqrt{\pi}}{h}.$$

expresses the probability that an error is numerically less than x. This may be written in the form

$$(4) \qquad P = \frac{2}{\sqrt{\pi}} \int_0^{hx} e^{-h^2x^2} d(hx),$$

and is called the probability integral.

As the number of errors of the magnitude x is proportional to the probability y, and as P in equation (4) is merely the summation of the probabilities of all errors between $-x$ and $+x$, the number of errors between these limits is also proportional to P. Now, P is the area of the probability curve between the limits $-x$ and $+x$, the whole area being unity. Hence the number of errors between two assigned limits ought to bear the same ratio to the whole number of errors as the value of P between these limits does to unity.

By the usual methods of the integral calculus the value of the probability integral corresponding to successive numerical values of hx may be computed.* A table of these values is given at the end of this volume (Table I.).

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* First put $hx = t$, then $\frac{2}{\sqrt{\pi}} \int_0^t e^{-t^2} dt$ is the integral to be evaluated. By developing e^{-t^2} into a series by Maclaurin's formula, the following results:

$$P = \frac{2}{\sqrt{\pi}} \left(t - \frac{t^3}{3} + \frac{1}{1.2} \cdot \frac{t^5}{5} - \frac{1}{1.2.3} \cdot \frac{t^7}{7} + \text{etc.} \right),$$

which is convenient for small values of t. For large values integrate by parts, thus

$$\int e^{-t^2} dt = -\frac{1}{2t} e^{-t^2} - \frac{1}{2} \int \frac{e^{-t^2}}{t^2} dt$$

$$= -\frac{1}{2t} e^{-t^2} + \frac{1}{2^2 t^3} e^{-t^2} + \frac{3}{2^2} \int \frac{e^{-t^2}}{t^4} dt.$$

To illustrate the use of this table, consider the case of $hx = 1.24$, for which $P = 0.9205$. Here 0.9205 is the probability that an error will be numerically less than $\frac{1.24}{h}$; or, in other words, if there be 10,000 observations, it is to be expected that in 9,205 of them the errors would lie between $-\frac{1.24}{h}$ and $+\frac{1.24}{h}$, and in the remaining 795 outside of these limits.

Comparison of Theory and Experience.

33. By means of Table I the theory employed in the deductions of equations (1), (2), (3), and (4) may be tested. To use the table it is necessary to know the value of the constant h. Granting for the present that it may be determined, the following examples will exemplify the accordance of theory and experience.

For the one hundred residual errors discussed in Art. 19, the value of h may be determined to be $\frac{1}{2''.236}$.

And since $\int_0^\infty e^{-t^2}dt = \frac{\sqrt{\pi}}{2}$, as shown in the preceding footnote,

$$\int_0^t e^{-t^2}dt = \frac{\sqrt{\pi}}{2} - \int_t^\infty e^{-t^2}dt,$$

from which $P = 1 - \frac{e^{-t^2}}{t\sqrt{\pi}}\left[1 - \frac{1}{2t^2} + \frac{1.3}{(2t^2)^2} - \frac{1.3.5}{(2t^2)^3} + \text{etc.}\right]$

From these two series the values of P can be found to any required degree of accuracy for all values of t or hx.

Then from the table the following values of P are taken:

$$\begin{array}{llll}
\text{for } x = 1''.0 & \text{with } hx = 0.447 & \text{the area } P = 0.473 \\
\text{for } x = 2.0 & \text{with } hx = 0.894 & \text{the area } P = 0.794 \\
\text{for } x = 3.0 & \text{with } hx = 1.341 & \text{the area } P = 0.942 \\
\text{for } x = 4.0 & \text{with } hx = 1.788 & \text{the area } P = 0.989 \\
\text{for } x = 5.0 & \text{with } hx = 2.235 & \text{the area } P = 0.998 \\
\text{for } x = \infty & \text{with } hx = \infty & \text{the area } P = 1.000
\end{array}$$

Now, these probabilities or areas P are proportional to the number of errors less than the corresponding values of x. Hence multiplying them by 100, the total number of errors, and subtracting each from that following, the number of theoretical errors between the successive values of x is found. The following is a comparison of the number of actual and theoretical errors:

Limits	Actual Errors.	Theoretical Errors.	Differences.
0''.0 and 1''.0	52	47	+5
1.0 and 2.0	30	32	−2
2.0 and 3.0	11	15	−4
3.0 and 4.0	4	5	−1
4.0 and 5.0	2	1	+1
5.0 and 6.0	1	0	+1
6.0 and ∞	0	0	0

The agreement between theory and experience, though not exact, is very satisfactory when the small number of observations is considered.

34. Numerous comparisons like the above have been made by different authors, and substantial agreement has always been found between the actual distribution of errors and the

theoretical distribution required by equations (2) and (4). The following is a comparison by Bessel of the errors of three hundred observations of the right ascensions of stars:

Limits.	Actual Errors.	Theoretical Errors.	Differences.
$0^s.0$ and $0^s.1$	114	107	+7
0.1 and 0.2	84	87	−3
0.2 and 0.3	53	57	−4
0.3 and 0.4	24	30	−6
0.4 and 0.5	14	13	+1
0.5 and 0.6	6	5	+1
0.6 and 0.7	3	1	+2
0.7 and 0.8	1	0	+1
0.8 and 0.9	1	0	+1
0.9 and ∞	0	0	0

The differences are here relatively smaller than in the previous case. And in general it is observed that the agreement between theory and experience is closer, the greater the number of errors or residuals considered in the comparison.

Whatever may be thought of the theoretical deductions of the law of probability of error, there can be no doubt but that its practical demonstration by experience is entirely satisfactory.

Remarks on the Fundamental Formulas.

35. The two equations of the probability curve,

$$(1) \qquad y = ke^{-h^2x^2},$$

$$(2) \qquad y = h.dx.\pi^{-\frac{1}{2}}e^{-h^2x^2},$$

are identical, and the former has already been discussed at

length. In the latter, dx for any special case is the interval between successive values of x. For instance, if observations of an angle be carried to tenths of seconds, dx is $0''.1$; if to hundredths of seconds, dx is $0''.01$; and if a continuous curve is considered, dx is the differential of x. As y is an abstract number, $h.dx$ must likewise be abstract, and hence h must be a quantity of the same kind as $\frac{1}{dx}$. The probability of the error 0 is $\frac{hdx}{\sqrt{\pi}}$; thus in measuring angles to hundredths of seconds, the probability that an error is $0''.00$ is $\frac{0''.01h}{\sqrt{\pi}}$. As this increases with h, the value of h may be regarded as a measure of the precision of the observations. Methods of determining h are given in Chap. IV.

36. The two probability integrals,

$$P = \frac{h}{\sqrt{\pi}} \int_{x'}^{x} e^{-h^2x^2} dx, \tag{3}$$

$$P = \frac{2}{\sqrt{\pi}} \int_{0}^{hx} e^{-h^2x^2} dhx, \tag{4}$$

are identical, except in their limits. The first gives the probability that an error will lie between any two limits x' and x; and the second, the probability that it lies between the limits $-x$ and $+x$, or that it is numerically less than x. The second is then a particular case of the first. Table I refers only to (4); and from it by simple addition or subtraction the probability can be found for any two assigned limits. For example, the probability that an error lies between $-2''.0$ and $+4''.0$ is the sum of the probabilities for the limits $0''.0$ to $2''.0$ and $0''.0$ to $4''.0$; and the probability that an error is between $+2''.0$ and $+4''.0$ is the difference of the probabilities of those limits.

The integral P is simply the summation of the values of y

between the assigned limits, or $P = \Sigma y$, as required by the principle of Art. 11 to express the probability of an error lying between those limits.

37. *Problems and Queries.*

1. Can cases be imagined where positive and negative errors are not equally probable?

2. An angle is measured to tenths of seconds by two observers, and the value of h for the first observer is double that for the second. Draw the two curves of probability of error.

3. Show that the arithmetical mean of two measurements is the only value that can be logically chosen to represent the quantity.

4. The reciprocal of h for the bullet-marks in Art. 18 is 2.33 feet. Compare the actual distribution of errors with the theoretical.

5. Draw a curve for each of the equations $y = ke^{-x^2}$ and $y = ke^{-4x^2}$, assuming a convenient value for k. Show that the value of k should have been taken different in the two equations.

6. Explain how the value of π might be determined by experiments with the help of equation (2).

CHAPTER III.

THE ADJUSTMENT OF OBSERVATIONS.

38. The Method of Least Squares comprises two tolerably distinct divisions. The first is the adjustment of observations, or the determination of the most probable values of observed quantities. The second is the investigation of the precision of the observations and of the adjusted results. This chapter contains the development of the rules and methods relating to the first division.

Weights of Observations.

39. Weights are numbers expressing the relative practical worth or value of observations. Thus, suppose a line to be measured twenty times with the same chain, ten measurements giving 934.2 feet, eight giving 934.0 feet, and two giving 934.6 feet; then the numbers 10, 8, and 2 are the weights of the respective observations 934.2, 934.0, and 934.6 feet. Or, since weights express only relative worth, the numbers 5, 4, and 1, or any other numbers proportional to 10, 8, and 2, may be taken as the weights. The observation 934.2 has cost five times as much as the observation 934.6, and for combination with other measurements it should be worth five times as much.

The weight of an observation expresses the number of standard observations of which it is the equivalent. Thus the average of n equally good direct measurements has a weight of n,

the weight of each single measurement being unity. And any observation having a weight of p * may be regarded as the equivalent of p observations of the weight unity, and as having a practical worth or value p times that of a single one. Hence the use of weights may be considered as a convenient method of abbreviation. Thus "934.2 with a weight of 10" expresses the same as the number 934.2 written down ten times, and regarded each time as a single observation.

40. A weighted observation is an observation multiplied by its weight. Thus if $M_1, M_2 \ldots M_n$ represent observations, and $p_1, p_2 \ldots p_n$ their respective weights, the products p_1M_1, $p_2M_2 \ldots p_nM_n$ represent weighted observations. If $x_1, x_2 \ldots x_n$ are the errors corresponding to $M_1, M_2 \ldots M_n$, the products p_1x_1, $p_2x_2 \ldots p_nx_n$ may be called weighted errors. As an error x is the difference between the true and measured value of the quantity observed, the product px cannot occur without implying that the corresponding observation M has a weight of p, and the same is true for the residual error v. Thus if there be two unknown quantities z_1 and z_2, and a measurement M be made upon $f(z_1, z_2)$, the residual error is

$$v = f(z_1, z_2) - M$$

if z_1 and z_2 denote the most probable values of the unknown quantities. Now, if the observation M be weighted with p, the residual is

$$pv = p.f(z_1, z_2) - pM.$$

Hence a weighted observation always implies a weighted residual, and vice versa.

The weights should be carefully distinguished from the measures of precision introduced in the last chapter. The former

* p is the initial of "pondus."

are relative abstract numbers, usually so selected as to be free from fractions, while the latter are absolute quantities. The relation between them will be shown in Art. 43.

The Principle of Least Squares.

41. The principle from which the term "Least Squares" arises is the following:

In measurements of equal precision the most probable values of observed quantities are those that render the sum of the squares of the residual errors a minimum.

To prove this, consider the general case of indirect observations, and let n equally good measurements be made upon functions of two unknown quantities z_1 and z_2. Let $M_1, M_2 \ldots M_n$ be the results of the measurements of the functions $f_1(z_1, z_2)$, $f_2(z_1, z_2) \ldots f_n(z_1, z_2)$. These measurements will not give exactly the true values of the functions, and the difference between the observed and true values will be small errors, $x_1, x_2 \ldots x_n$, or

$$f_1(z_1, z_2) - M_1 = x_1, \quad f_2(z_1, z_2) - M_2 = x_2 \ldots f_n(z_1, z_2) - M_n = x_n.$$

The respective probabilities of these errors are by the fundamental law (1)

$$y_1 = ke^{-h^2x_1^2}, \quad y_2 = ke^{-h^2x_2^2} \ldots y_n = ke^{-h^2x_n^2},$$

h being the same in all, since the observations are of equal precision. Now, by Art. 12, the probability of the compound event of committing the system of independent errors $x_1, x_2 \ldots x_n$ is the product of these separate probabilities, or

$$P' = c^n e^{-h^2(x_1^2 + x_2^2 + x_3^2 + \ldots + x_n^2)}.$$

Each of these errors is a function of the quantities z_1 and z_2, which are to be determined. Different values of z_1 and z_2 will

give different values for P'. The most probable system of errors will be that for which P' is a maximum (Art. 13), and the most probable values of z_1 and z_2 will correspond to the most probable system of errors. The probability P' will be a maximum when the exponent of e is a maximum; that is when

$$x_1^2 + x_2^2 + x_3^2 + \ldots + x_n^2 = \text{a minimum.}$$

Hence the most probable system of values for z_1 and z_2 is that which renders the sum $x_1^2 + x_2^2 + x_3^2 + \ldots + x_n^2$ a minimum, and the fundamental principle of Least Squares is thus proved.

The errors $x_1, x_2 \ldots x_n$ have been thus far regarded as the true errors of the observations. As soon, however, as they are required to satisfy the condition that the sum of their squares is a minimum, they become residual errors (Art. 8), so that the condition for the most probable values of z_1 and z_2 is really

$$(5) \qquad v_1^2 + v_2^2 + v_3^2 + \ldots + v_n^2 = \text{a minimum};$$

that is to say, if z_1 and z_2 be the most probable values, the computed residuals

$$f_1(z_1, z_2) - M_1 = v_1, \;\; f_2(z_1, z_2) - M_2 = v_2 \ldots f_n(z_1, z_2) - M = v_n$$

will be those that satisfy the condition for a minimum.

The above reasoning evidently applies to any number of unknown quantities as well as to two.

42. The more general case of the Method of Least Squares, however, is that when the observations have different degrees of precision, or different weights. In that event the general principle is the following:—

In measurements of unequal weight the most probable values of observed quantities are those that render the sum of the weighted squares of the residual errors a minimum.

As before, let n observations, $M_1, M_2 \ldots M_n$, be made upon functions of two unknown quantities, z_1 and z_2; and let $p_1, p_2 \ldots p_n$ be the respective weights of $M_1, M_2 \ldots M_n$. The differences between the observations and the true values of the functions are errors, $x_1, x_2 \ldots x_n$; and the respective probabilities of these errors are

$$y_1 = k_1 e^{-h_1^2 x_1^2}, \quad y_2 = k_2 e^{-h_2^2 x_2^2} \ldots y_n = k_n e^{-h_n^2 x_n^2},$$

in which k and h are different for each observation. The probability of the system of independent errors, $x_1, x_2 \ldots x_n$, then, is

$$P' = k_1 k_2 \ldots k_n e^{-(h_1^2 x_1^2 + h_2^2 x_2^2 + \ldots + h_n^2 x_n^2)};$$

and the most probable system of values is that for which P' is a maximum, or that which renders

$$h_1^2 x_1^2 + h_2^2 x_2^2 + \ldots + h_n^2 x_n^2 = \text{a minimum}.$$

The values of $x_1, x_2 \ldots x_n$, derived from this condition, are the residual errors, $v_1, v_2 \ldots v_n$; so that it will be well to write at once

$$h_1^2 v_1^2 + h_2^2 v_2^2 + \ldots + h_n^2 v_n^2 = \text{a minimum}.$$

This expression may be divided by h^2; h being a constant standard measure of precision so selected, that

$$h_1^2 = p_1 h^2, \quad h_2^2 = p_2 h^2 \ldots h_n^2 = p_n h^2,$$

where $p_1, p_2 \ldots p_n$ are whole numbers, which are the weights of the observations $M_1, M_2 \ldots M_n$.* Then it becomes

$$(6) \qquad p_1 v_1^2 + p_2 v_2^2 + \ldots + p_n v_n^2 = \text{a minimum};$$

* To show that these numbers are the weights of $M_1, M_2 \ldots M_n$, consider that the condition for the minimum will be fulfilled when

$$h_1^2 v_1 \frac{dv_1}{dz_1} + h_2^2 v_2 \frac{dv_2}{dz_1} + \ldots + h_n^2 v_n \frac{dv_n}{dz_1} = 0,$$

$$h_1^2 v_1 \frac{dv_1}{dz_2} + h_2^2 v_2 \frac{dv_2}{dz_2} + \ldots + h_n^2 v_n \frac{dv_n}{dz_2} = 0,$$

which is the principle that was to be proved. The term "weighted square" means simply v^2 multiplied by the weight p, or the product pv^2.

The conditions expressed by (5) and (6) are the fundamental ones for the establishment of the practical rules for the adjustment of independent observations. If the observations are of equal weight, the general condition (6) reduces to the special one (5).

43. It is here seen that the squares of the measures of precision of observations are proportional to the weights, or that

$$(7) \qquad h_1^2 : h_2^2 : h^2 :: p_1 : p_2 : p.$$

The measure of precision is never used in the practical application of the Method of Least Squares, while weights are constantly employed. The quantity h, however, is very convenient in the theoretical discussions, and will be needed often in the next chapter: h represents an absolute quantity, while p denotes always an abstract number.

Direct Observations on a Single Quantity.

44. When the observations are of equal precision, and made directly on the quantity whose value is sought, it is universally recognized that the arithmetical mean is the most probable

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which, after dividing by the standard h^2, become

$$p_1 v_1 \frac{dv_1}{dz_1} + p_2 v_2 \frac{dv_2}{dz_1} + \ldots + p_n v_n \frac{dv_n}{dz_1} = 0,$$

$$p_1 v_1 \frac{dv_1}{dz_2} + p_2 v_2 \frac{dv_2}{dz_2} + \ldots + p_n v_n \frac{dv_n}{dz_2} = 0.$$

Here the residual v_1 is repeated p_1 times, v_2 is repeated p_2 times, and v_n is repeated p_n times, and hence $p_1, p_2 \ldots p_n$ are the weights of the corresponding observations $M_1, M_2 \ldots M_n$ (Art. 40).

value of the quantity. This may be also shown from the fundamental principle of Least Squares in the following manner:

Let $M_1, M_2 \ldots M_n$ denote the direct observations which are all of equal weight or precision. Let z be the most probable value which is to be determined. Then the residual errors are

$$z - M_1, \quad z - M_2 \ldots z - M_n,$$

and from the fundamental principle (5)

$$(z - M_1)^2 + (z - M_2)^2 + \ldots + (z - M_n)^2 = \text{a minimum.}$$

To apply the usual method for maxima and minima, place the first derivative of this expression equal to zero, thus

$$2(z - M_1) + 2(z - M_2) + \ldots + 2(z - M_n) = 0.$$

Dividing this by 2, and solving for z, gives

$$(8) \qquad z = \frac{M_1 + M_2 + M_3 + \ldots + M_n}{n};$$

that is, the most probable value z is the arithmetical mean of the n observations.

The adjustment of direct observations of equal weight on the same quantity is hence effected by taking the arithmetical mean of the observations.

45. When the measurements of a quantity are of unequal weight or precision, the arithmetical mean does not apply. Here the more general principle (6) will furnish the proper rule to employ. Let the measurements be $M_1, M_2 \ldots M_n$, having the weights $p_1, p_2 \ldots p_n$. Then, if z be the most probable value of the observed quantity, the expression (6) becomes

$$p_1(z - M_1)^2 + p_2(z - M_2)^2 + \ldots + p_n(z - M_n)^2 = \text{a minimum.}$$

Placing the first derivative of this equal to zero gives

$$p_1(z - M_1) + p_2(z - M_2) + \ldots + p_n(z - M_n) = 0,$$

the solution of which is

$$z = \frac{p_1M_1 + p_2M_2 + \ldots + p_nM_n}{p_1 + p_2 + \ldots + p_n}; \tag{9}$$

that is, the most probable value of the unknown quantity z is obtained by multiplying each observation by its weight, and dividing the sum of the products by the sum of the weights. In order to distinguish this process from that of the arithmetical mean, it is sometimes called the general mean, or the weighted mean.

Granting that the arithmetical mean gives the most probable value for observations of equal weight, the general mean (9) for observations of unequal weight may be readily deduced from the definitions of the word "weight" in Art. 39.

The adjustment of direct observations of unequal weight on the same quantity is hence effected by taking the general mean of the observations.

Independent Observations of Equal Weight.

46. The general case of independent observations comprises several unknown quantities whose values are to be determined from either direct or indirect measurements made upon them.

An "observation equation" is an equation connecting the observation with the quantities sought. Thus, if M be a measurement of $f(z_1, z_2)$, the equation $M = f(z_1, z_2)$ is an observation equation. The number of these equations is the same as the number of observations, and generally greater than the number of unknown quantities to be determined. Hence, in general,

no system of values can be found which will exactly satisfy the observation equations. They may, however, be approximately satisfied by many systems of values; and the problem is to determine that system which is the most probable, or which has the maximum probability (Art. 13).

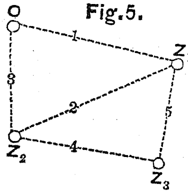


Fig. 5.

To illustrate, consider the following practical case. Let O represent a given bench-mark, and Z_1, Z_2, Z_3, three points whose elevations above O are to be determined. Let five lines of levels be run between these points, giving the following results:

Observation 1. Z_1 above $O = 10$ feet.
Observation 2. Z_2 above $Z_1 = 7$ feet.
Observation 3. Z_2 above $O = 18$ feet.
Observation 4. Z_2 above $Z_3 = 9$ feet.
Observation 5. Z_3 below $Z_1 = 2$ feet.

If the elevations of the points Z_1, Z_2, and Z_3, be designated by z_1 z_2, and z_3, the following observation equations may be written:

$$\begin{aligned} z_1 &= 10, \\ z_2 - z_1 &= 7, \\ z_2 &= 18, \\ z_2 - z_3 &= 9, \\ z_1 - z_3 &= 2, \end{aligned}$$

each one of which is an approximation to the truth, but all of which cannot be correct. The number of these equations is five, the number of the unknown quantities is three; and hence an exact solution cannot be made. The problem is to find the most probable values of z_1, z_2, and z_3.

The observation equations may be algebraic expressions of the first, second, or higher degrees; or they may contain circular or logarithmic functions. Usually, however, they are of the first degree, or linear, and these alone will be considered in the

body of this work. In Art. 140 is given a method by which non-linear equations, should they occur, may always be reduced to linear ones.

47. Consider first the case of observations of equal precision or of equal weight. Let there be q unknown quantities z_1, $z_2 \ldots z_q$, and let the equations between them and the measured quantities be of the form

$$az_1 + bz_2 + \ldots + lz_q = M,$$

in which $a, b \ldots l$ are constants given by theory and absolutely known, and M the measured quantity. For each observation, there will be a similar equation, and, in all, the following n approximate observation equations:

$$\begin{aligned}
a_1z_1 + b_1z_2 + \ldots + l_1z_q &= M_1,\\
a_2z_1 + b_2z_2 + \ldots + l_2z_q &= M_2,\\
a_3z_1 + b_3z_2 + \ldots + l_3z_q &= M_3,\\
\cdots\cdots\cdots&\cdots\\
a_nz_1 + b_nz_2 + \ldots + l_nz_q &= M_n,
\end{aligned}$$

the first of which arises from the first observation, the second from the second, and the last from the n^{th}.

Now, as the number of these observation equations is greater than that of the unknown quantities, they will not be exactly satisfied for any system of values that may be deduced. The best that can be done is to find, from the fundamental principle of Least Squares, the most probable system. Let $z_1, z_2 \ldots z_q$ denote the most probable values, then, if these be substituted in the observation equations, they will not reduce exactly to zero, but leave small residuals, $v_1, v_2 \ldots v_n$; thus strictly

$$\begin{aligned}
a_1z_1 + b_1z_2 + \ldots + l_1z_q - M_1 &= v_1,\\
a_2z_1 + b_2z_2 + \ldots + l_2z_q - M_2 &= v_2,\\
\cdots\cdots\cdots&\cdots\\
a_nz_1 + b_nz_2 + \ldots + l_nz_q - M_n &= v_n.
\end{aligned}$$

The fundamental principle established in Art. 41 is, that the most probable values, $z_1, z_2 \ldots z_q$, are those that render

$$v_1^2 + v_2^2 + v_3^2 + \ldots + v_n^2 = \text{a minimum.}$$

Consider first what is the most probable value of the unknown quantity z_1, and denote the terms in the above equations independent of z_1 by the letters N_1, N_2, N_3, etc. Then they become

$$\begin{aligned} a_1z_1 + N_1 &= v_1, \\ a_2z_1 + N_2 &= v_2, \\ \cdot \quad \cdot \quad &\cdot \quad \cdot \quad \cdot \\ a_nz_1 + N_n &= v_n. \end{aligned}$$

Squaring both terms of each of these equations, and adding the results, gives

$$(a_1z_1 + N_1)^2 + (a_2z_1 + N_2)^2 + \ldots + (a_nz_1 + N_n)^2 = v_1^2 + v_2^2 + \ldots + v_n^2.$$

In order to make this sum a minimum, its first derivative must be put equal to zero, giving

$$c_1(a_1z_1 + N_1) + a_2(a_2z_1 + N_2) + \ldots + a_n(a_nz_1 + N_n) = 0;$$

and this is the condition for the most probable value of z_1. In like manner a similar condition may be found for each of the other unknown quantities. The number of these conditions, or "normal equations" as they are called, will be the same as that of the unknown quantities, and their solution will furnish the most probable values of $z_1, z_2 \ldots z_q$.

48. The following is, hence, the method for the adjustment of independent indirect observations of equal weight:

For each observation write an observation equation. Form a normal equation for z_1 by multiplying each of the observation equations by the co-efficient of z_1 in that equation, and adding

the results. And, for each unknown quantity, form a normal equation by multiplying each observation equation by the coefficient of that unknown quantity in that equation, and adding the results. The solution of these normal equations will furnish the most probable values of the unknown quantities.

For example, let the five observation equations derived from the five observations of Art. 46 be considered, namely,

$$\begin{aligned}
z_1 \qquad\qquad &= 10, \\
-z_1 + z_2 \qquad &= 7, \\
z_2 \qquad &= 18, \\
z_2 - z_3 &= 9, \\
z_1 \qquad - z_3 &= 2.
\end{aligned}$$

To form the normal equation for z_1 the first observation equation is multiplied by $+1$, the second by -1, the third by 0, the fourth by 0, and the fifth by $+1$; the addition of the products then gives

$$3z_1 - z_2 - z_3 = 5.$$

The normal equation for z_2 is formed by multiplying the first observation equation by 0, the second by $+1$, the third by $+1$, the fourth by $+1$, and the fifth by 0; the sum of the products being

$$-z_1 + 3z_2 - z_3 = 34.$$

The normal equation for z_3 is formed by multiplying the first, second, and third observation equations by 0, and the fourth and fifth by -1, the addition of which gives

$$-z_1 - z_2 + 2z_3 = -11.$$

These three normal equations contain three unknown quantities, and their solution gives

$$z_1 = +10\tfrac{3}{8}, \quad z_2 = +17\tfrac{5}{8}, \quad z_3 = +8\tfrac{1}{2},$$

which are the most probable values that can be obtained from the five observations. If now these values be substituted in the observation equations there will be found the five residuals,

$$v_1 = +\tfrac{3}{8}, \quad v_2 = +\tfrac{1}{4}, \quad v_3 = -\tfrac{3}{8}, \quad v_4 = +\tfrac{1}{8}, \quad v_5 = -\tfrac{1}{8},$$

and the sum of the squares of these is $\frac{3}{8}$. Of all the possible values that might be assigned to z_1, z_2, z_3, those above found give the minimum sum of squares of residual errors.

As a second example, let three observations on the two quantities z_1 and z_2 give the observation equations

$$\begin{aligned} 3z_1 - 5z_2 &= +12.4, \\ -2z_1 + 4z_2 &= -10.2, \\ z_1 - 2z_2 &= +\ 8.0. \end{aligned}$$

To form the normal equation for z_1 the observation equations are multiplied by $+3$, -2, and $+1$, respectively, and the results added. To form the normal equation for z_2 the multipliers are -5, $+4$, and -2, respectively. The two normal equations thus are

$$\begin{aligned} 14z_1 - 25z_2 &= +\ 65.6, \\ -25z_1 + 45z_2 &= -118.8, \end{aligned}$$

and the solution of these gives the most probable values $z_1 = -3.60$ and $z_2 = -4.64$.

49. In order to put the above method for the formation of normal equations into algebraic language, let there be n observations upon q unknown quantities which lead to the following observation equations:

$$(10) \qquad \begin{aligned} a_1z_1 + b_1z_2 + c_1z_3 + \ldots + l_1z_q &= M_1, \\ a_2z_1 + b_2z_2 + c_2z_3 + \ldots + l_2z_q &= M_2, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \quad \cdot \quad \cdot \\ a_nz_1 + b_nz_2 + c_nz_3 + \ldots + l_nz_q &= M_n. \end{aligned}$$

The normal equation for z_1 is formed by multiplying the first of these by a_1, the second by a_2, the last by a_n, and adding the products, thus giving

$$(a_1^2 + a_2^2 + \ldots + a_n^2)z_1 + (a_1b_1 + a_2b_2 + \ldots + a_nb_n)z_2 + \ldots$$
$$= (a_1M_1 + a_2M_2 + \ldots + a_nM_n);$$

and in like manner a normal equation for each of the other unknown quantities may be written. To simplify the expression of these equations, let the following abbreviations for summation be introduced:

$$\begin{aligned}
[aa] &= a_1^2 + a_2^2 + a_3^2 + \ldots + a_n^2,\\
[ab] &= a_1b_1 + a_2b_2 + a_3b_3 + \ldots + a_nb_n,\\
[al] &= a_1l_1 + a_2l_2 + a_3l_3 + \ldots + a_nl_n,\\
[bb] &= b_1^2 + b_2^2 + b_3^2 + \ldots + b_n^2,\\
[aM] &= a_1M_1 + a_2M_2 + a_3M_3 + \ldots + a_nM_n,
\end{aligned}$$

and then the normal equations may be thus written:

$$\begin{aligned}
&[aa]z_1 + [ab]z_2 + [ac]z_3 + \ldots + [al]z_q = [aM],\\
&[ba]z_1 + [bb]z_2 + [bc]z_3 + \ldots + [bl]z_q = [bM],\\
(11)\quad &[ca]z_1 + [cb]z_2 + [cc]z_3 + \ldots + [cl]z_q = [cM],\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
&[la]z_1 + [lb]z_2 + [lc]z_3 + \ldots + [ll]z_q = [lM].
\end{aligned}$$

The co-efficients of the unknown quantities in these normal equations present a curious symmetry; those of the first horizontal row being the same as those of the first vertical column, those of the second row the same as those of the second column, and so on. This is due to the fact that $[ba]$ is the same as $[ab]$, $[ca]$ the same as $[ac]$, ... and $[la]$ the same as $[al]$.

The notation for summation here indicated is that first used by Gauss and since generally employed in works on the Method of Least Squares in writing normal equations. The notation Σa^2, Σab, used by a few writers, and in former editions of this

book, has the same meaning as $[aa]$, $[ab]$. The sum of the squares of the residual errors may be written either Σv^2 or $[vv]$, and in this book the former will be employed as it more readily calls to mind its name.

50. Hence the method of adjustment of indirect observations of equal weight is to write for the n observations the n observation equations (10), then to form the q normal equations (11), and their solution will furnish the most probable values of the unknown quantities. Numerous examples of the application of this method will be found in Chap. VII.

As a simple illustration let three observation equations be

$$\begin{aligned} 4z_1 - 2z_2 &= +6.1, \\ 5z_1 + 2z_2 &= +3.8, \\ 3z_1 - 3z_2 &= -0.9. \end{aligned}$$

Here $a_1 = +4$, $a_2 = +5$, $a_3 = +3$, $b_1 = -2$, $b_2 = +2$, $b_3 = -3$, $M_1 = +6.1$, $M_2 = +3.8$, $M_3 = -0.9$. The formation of the sums is now made, carefully regarding the signs of the co-efficients; thus,

$$\begin{aligned} [aa] &= +\ 4^2 + 5^2 + 3^2 = +50.0, \\ [ab] &= -\ 8 + 10 - 9 = -\ 7.0, \\ [aM] &= +24.4 + 19.0 - 2.7 = +40.7, \\ [bb] &= 2^2 + 2^2 + 3^2 = +17.0, \\ [bM] &= -12.2 + 7.6 + 2.7 = -\ 1.9. \end{aligned}$$

Here $[ba]$ need not be computed, as its value is the same as $[ab]$; thus the two normal equations are

$$\begin{aligned} +50z_1 - 7z_2 &= +40.7, \\ -7z_1 + 17z_2 &= -\ 1.9, \end{aligned}$$

the solution of which gives $z_1 = +0.8472$ and $z_2 = +0.2371$ as the most probable values correct to the fourth decimal place.

Independent Observations of Unequal Weight.

51. The more usual case in practice is where the observations have unequal weights. As weights are merely numbers denoting repetition, it is plain that if each observation equation be written as many times as indicated by its weight, the reasoning of Art. 47 and the rule of Art. 48 applies directly to the determination of the probable values of the unknown quantities. Instead, however, of writing an observation equation as many times as indicated by its weight, it will be sufficient to multiply it by its weight when forming the other products.

52. The following rule may hence be stated for the adjustment of independent observations of unequal weight upon several related quantities:

For each weighted observation write an observation equation, noting its weight. Form a normal equation for z_1 by multiplying each equation by the co-efficient of z_1 in that equation, and also by its weight, and adding the products. In like manner form a normal equation for each of the other unknown quantities by multiplying each observation equation by the coefficient of that unknown quantity in that equation, and also by its weight, and adding the results. The solution of these normal equations will furnish the most probable values of the unknown quantities.

For example, let three observations upon two unknown quantities give the three observation equations,

$$\begin{aligned} -2z_1 + 3z_2 &= +6, \quad \text{weight } 3, \\ +2z_1 \qquad &= +3, \quad \text{weight } 7, \\ -3z_2 &= +5, \quad \text{weight } 2. \end{aligned}$$

To form the normal equation for z_1 the first equation is multiplied by the co-efficient -2 and by the weight 3, that is, by

-6; the second is multiplied by $+2$ and 7, that is, by $+14$; the third is multiplied by 0 and 2, that is, by 0; the addition of the products gives

$$+40z_1 - 18z_2 = +6.$$

To form the normal equation for z_2, the first equation is multiplied by $+3$ and by 3, that is, $+9$; the second by 0, and the third by -6; the sum of products being

$$-18z_1 + 45z_2 = +24.$$

The solution of these two normal equations gives $z_1 = +0.475$ and $z_2 = +0.724$ as the most probable values of the two quantities which were indirectly observed.

53. In order to put this method into an algebraic algorithm and at the same time review the general reasoning, let M_1, $M_2, \ldots M_n$ be the results of the n observations which have been made to determine the values of the q quantities $z_1, z_2, \ldots z_q$. As before, let each observation be represented by an observation equation, thus:

$$
(12)\qquad
\begin{array}{ll}
a_1z_1 + b_1z_2 + \ldots + l_1z_q = M_1 & \text{with weight } p_1, \\
a_2z_1 + b_2z_2 + \ldots + l_2z_q = M_2 & \text{with weight } p_2, \\
\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot & \cdot\quad\cdot\quad\cdot\quad\cdot \\
a_nz_1 + b_nz_2 + \ldots + l_nz_q = M_n & \text{with weight } p_n.
\end{array}
$$

Now, if $z_1, z_2 \ldots z_q$ denote the most probable values of the quantities sought, and these values be substituted in (12), these equations will not reduce to zero, but leave small residuals, v_1, $v_2 \ldots v_n$. Thus strictly,

$$
\begin{array}{ll}
a_1z_1 + b_1z_2 + \ldots + l_1z_q - M_1 = v_1 & \text{with weight } p_1, \\
a_2z_1 + b_2z_2 + \ldots + l_2z_q - M_2 = v_2 & \text{with weight } p_2, \\
\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot & \cdot\quad\cdot\quad\cdot\quad\cdot \\
a_nz_1 + b_nz_2 + \ldots + l_nz_q - M_n = v_n & \text{with weight } p_n,
\end{array}
$$

which may be called residual equations.

Now, according to the general principle (6), the most probable values of the q unknown quantities $z_1, z_2 \ldots z_q$ are those that render the expression

$$p_1v_1^2 + p_2v_2^2 + \ldots + p_nv_n^2 = \text{a minimum.}$$

To abbreviate, designate this quantity by Σpv^2. Remembering that $v_1^2, v_2^2 \ldots v_n^2$ are functions of $z_1, z_2 \ldots z_q$, it is plain that the derivative of Σpv^2 with reference to each variable must be zero, and that hence there are the following q conditions for the minimum:

$$p_1v_1\frac{dv_1}{dz_1} + p_2v_2\frac{dv_2}{dz_1} + \ldots + p_nv_n\frac{dv_n}{dz_1} = 0,$$
$$p_1v_1\frac{dv_1}{dz_2} + p_2v_2\frac{dv_2}{dz_2} + \ldots + p_nv_n\frac{dv_n}{dz_2} = 0,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$p_1v_1\frac{dv_1}{dz_q} + p_2v_2\frac{dv_2}{dz_q} + \ldots + p_nv_n\frac{dv_n}{dz_q} = 0.$$

The values of the differential co-efficients in these conditions are readily found by taking the derivatives of the residual equations with reference to each variable, thus:

$$\frac{dv_1}{dz_1} = a_1, \quad \frac{dv_2}{dz_1} = a_2, \quad \frac{dv_1}{dz_2} = b_1, \text{ etc.};$$

and the conditions then become

$$p_1a_1v_1 + p_2a_2v_2 + \ldots + p_na_nv_n = 0,$$
$$p_1b_1v_1 + p_2b_2v_2 + \ldots + p_nb_nv_n = 0,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$p_1l_1v_1 + p_2l_2v_2 + \ldots + p_nl_nv_n = 0,$$

which are as many in number as there are unknown quantities $z_1, z_2 \ldots z_q$. If in these the values of $v_1, v_2 \ldots v_n$ be replaced

from the residual equations, the final normal equations will result. As before, the expression of the normal equations may be abbreviated by using the square bracketed notation for summation, namely,

$$
\begin{aligned}
[paa] &= p_1a_1^2 + p_2a_2^2 + \ldots + p_na_n^2,\\
[pab] &= p_1a_1b_1 + p_2a_2b_2 + \ldots + p_na_nb_n,\\
[paM] &= p_1a_1M_1 + p_2a_2M_2 + \ldots + p_na_nM_n, \text{ etc.,}
\end{aligned}
$$

and thus the normal equations are

$$
(13)\quad
\begin{aligned}
&[paa]z_1 + [pab]z_2 + \ldots + [pal]z_q = [paM],\\
&[pba]z_1 + [pbb]z_2 + \ldots + [pbl]z_q = [pbM],\\
&\ldots\ldots\ldots\ldots\ldots\ldots\\
&[pla]z_1 + [plb]z_2 + \ldots + [pll]z_q = [plM],
\end{aligned}
$$

by whose solution the most probable values of $z_1, z_2 \ldots z_q$ may be found. The co-efficients in these equations show the same symmetry as those in Art. 48, since, as before, $[pba]$, $[plb]$, etc., are the same as $[pab]$, $[pbl]$, etc.

54. Thus, if there be n observations for determining q unknown quantities, the most probable values of the unknown quantities are obtained by writing n observation equations as in (12), and forming the q normal equations as in (13); then the solution of these normal equations will furnish the most probable values of the q unknown quantities. In the most common cases the co-efficients in the observation equations (12) are $+1$, -1, or 0, and, in the formation of the co-efficients of the normal equations, the signs must be carefully regarded. Many examples of adjustment by this method are given in Chap. VII.

As a simple illustration let there be given the following four weighted observation equations upon the two quantities z_1 and z_2:

$$
\begin{array}{llll}
\text{No. 1.} & +z_1 \qquad = 0, & \text{weight } p_1 = & 8, \\
2. & \qquad + \; z_2 = 0, & p_2 = & 10, \\
3. & +z_1 + 2z_2 = +0.25, & p_3 = & 1, \\
4. & +z_1 - 3z_2 = -0.92, & p_4 = & 5,
\end{array}
$$

These co-efficients and weights, arranged in tabular form, are

No.	a	b	M	p
1.	+1	0	0	8
2.	0	+1	0	10
3.	+1	+2	+0.25	1
4.	+1	−3	−0.92	5

The products paa, pab, etc., are now formed as below, and their summation furnishes the co-efficients for the two normal equations; thus,

No.	paa	pab	paM	pbb	pbM
1.	+ 8	0	0	0	0
2.	0	0	0	+10	0
3.	+ 1	+ 2	+0.25	+ 4	+ 0.50
4.	+ 5	−15	−4.60	+45	+13.80
	+14	−13	−4.35	+59	+14.30

Here $[pba]$ is the same as $[pab]$ and need not be computed.

The normal equations therefore are

$$
\begin{aligned}
+14z_1 - 13z_2 &= - \;\; 4.35, \\
-13z_1 + 59z_2 &= + 14.30,
\end{aligned}
$$

the solution of which gives $z_1 = -0.102$ and $z_2 = +0.225$ as the most probable values that can be derived from the four observations. If these be substituted in the observation equations the adjusted values of the four observations are found to be -0.102, $+0.225$, $+0.348$, and -0.777.

In the formation of the co-efficients of normal equations tables of squares, multiplication tables, and calculating machines will often be found very useful. The method of using the table of squares at the end of this volume for the formation of the products ab, ac, etc., is explained in Art. 172, and a method for checking the correctness of the co-efficients $[ab]$, $[ac]$, etc., is given in Art. 142.

Solution of Normal Equations.

55. The normal equations which arise in the adjustment of observations may be solved by any algebraic process. When the co-efficients consist of several digits, or when the number of unknowns is greater than three, it is desirable to follow a method by which checks may be constantly obtained upon the accuracy of the numerical work. Such a method, devised by Gauss, is presented in Chap. X.

When the number of unknown quantities is two, the observation equations furnish the two normal equations

$$[aa]z_1 + [ab]z_2 = [aM],$$
$$[ab]z_1 + [bb]z_2 = [bM],$$

the solution of which may be directly effected by the formulas

$$z_1 = \frac{[bb][aM] - [ab][bM]}{[aa][bb] - [ab]^2},$$
$$z_2 = \frac{[aa][bM] - [ab][aM]}{[aa][bb] - [ab]^2},$$

while checks upon the numerical work may be obtained by substituting the computed values in the given normal equations which should be exactly or closely satisfied.

When logarithms are used it will generally be advantageous to write the formulas thus

$$z_1 = \frac{[bb][aM]/[ab] - [bM]}{[aa][bb]/[ab] - [ab]},$$

$$z_2 = \frac{[aa][bM]/[ab] - [aM]}{[aa][bb]/[ab] - [ab]},$$

as then the table need be entered only three times in finding the numbers corresponding to two terms in the numerator and one in the denominator, whereas by the former formulas six entries are required.

As an example let the two normal equations be

$$90.07z_1 + 404.56z_2 = 295.99,$$
$$404.56z_1 + 1934.10z_2 = 1306.90.$$

Here, by the use of either numbers or logarithms, the solution gives the values

$$z_1 = + 4.1527, \qquad z_2 = - 0.1929,$$

which, substituted in the normal equations, reduce the first to $+0.004 = 0$ and the second to $+0.028 = 0$. The first is satisfied as closely as the data admit, while the error in the second can be reduced, if deemed necessary, by carrying the values of z_1 and z_2 to five decimal places.

When the number of unknown quantities is three, general formulas for solution are best derived in the determinant form given in Art. 140. This determinant method is easily remembered and may be advantageously used for the case of two unknown quantities.

Conditioned Observations.

56. Thus far it has been considered that the quantities to be determined by observation were independent of each other. Although they have been related to each other through the

observation equations, and have been required to satisfy approximately those equations, they have been so far independent, that any one unknown quantity might be supposed to vary without affecting the values of the others. All systems of values of the unknown quantities have been regarded equally possible, and the methods above developed show how to determine the most probable system.

In the class of observations now to be discussed, all systems of values are not equally possible, owing to the existence of conditions which must be exactly satisfied. Thus, having measured two angles of a triangle, the adjusted value of one is entirely independent of that of the other; but, if the third angle be measured, the three angles are subject to the rigorous geometrical condition that their sum must be exactly 180°. In conditioned observations there are, hence, two classes of equations, observation equations and conditional equations; the number of the first being generally greater than the number of unknown quantities, and that of the latter always less.*

57. Designate the number of observation equations by n, the number of unknown quantities by q, and the number of conditional equations by n'. If no conditional equations existed, the principle of Least Squares (6) would require that the adjusted system of values should be the most probable for the n independent observations. The n' conditional equations, being less in number than the q unknown quantities, may be satisfied in various ways; and, further, the final adjusted system of values must exactly satisfy them. Hence it must be concluded, that, of all the systems of values which exactly satisfy the n' conditional equations, that one is to be chosen, which in the n

* In most books upon this subject, the term "equations of condition" is applied indiscriminately to both of these very distinct classes, and is a cause of some perplexity to the student. The excellent distinction of the Germans, "Beobachtungsgleichung" and "Bedingungsgleichung," ought certainly to come into use.

observation equations makes the sum of the weighted squares of the residuals a minimum.

The problem of conditioned observations may be, then, reduced to that of independent ones by finding from the n' conditional equations the values of n' unknown quantities in terms of the remaining $q - n'$ quantities, and substituting them in the n observation equations. There will thus result n independent observations upon $q - n'$ quantities. From these the normal equations are to be formed, and the most probable values of the $q - n'$ quantities deduced. Substituting these values in the n' conditional equations, the remaining n' quantities become known. Thus the q quantities exactly satisfy the conditional equations, and at the same time are the most probable values for the observation equations. This, therefore, is a general solution of the problem.

For example, consider the measurement of the three angles of a plane triangle. Let z_1, z_2, and z_3 be the most probable values of the angles, and let the observation equations be

$$z_1 = M_1, \quad z_2 = M_2, \quad z_3 = M_3,$$

which are subject to the rigorous condition

$$z_1 + z_2 + z_3 = 180°.$$

From the conditional equation take the value of z_3, and substitute it in the observation equations, giving

$$z_1 = M_1, \quad z_2 = M_2, \quad z_1 + z_2 = 180° - M_3.$$

The most probable values of z_1 and z_2 may be now obtained by the method of Art. 47, since the three observation equations are independent. Then the most probable value of z_3 is $180° - z_1 - z_2$.

58. Although the above method is perfectly general, and very simple in theory, it gives rise in practice to tedious computa-

tions whenever the number of conditional equations is large. The process generally employed is the "method of correlatives," due to Gauss, which will now be explained; the conditional equations being considered as of the first degree, or linear, and the number of observations being the same as that of the quantities to be determined, or $n = q$.

Consider q unknown quantities connected by the n' rigorous conditions,

$$(14) \qquad \begin{aligned} \alpha_0 + \alpha_1 z_1 + \alpha_2 z_2 + \ldots + \alpha_q z_q &= 0, \\ \beta_0 + \beta_1 z_1 + \beta_2 z_2 + \ldots + \beta_q z_q &= 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots& \\ \lambda_0 + \lambda_1 z_1 + \lambda_2 z_2 + \ldots + \lambda_q z_q &= 0. \end{aligned}$$

Let $M_1, M_2 \ldots M_q$ be the values found by the observations for $z_1, z_2 \ldots z_q$. If these be inserted in the conditional equations, they will not reduce to zero, but leave small discrepancies, $d_1, d_2 \ldots d_{n'}$, thus:

$$\begin{aligned} \alpha_0 + \alpha_1 M_1 + \alpha_2 M_2 + \ldots + \alpha_q M_q &= d_1, \\ \beta_0 + \beta_1 M_1 + \beta_2 M_2 + \ldots + \beta_q M_q &= d_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots& \\ \lambda_0 + \lambda_1 M_1 + \lambda_2 M_2 + \ldots + \lambda_q M_q &= d_{n'}. \end{aligned}$$

Let $v_1, v_2 \ldots v_q$ be corrections, which when applied to M_1, $M_2 \ldots M_q$, will render them the most probable values, and cause the discrepancies to disappear; thus, if $z_1, z_2 \ldots z_q$ be the most probable values,

$$z_1 = M_1 + v_1, \quad z_2 = M_2 + v_2 \ldots z_q = M_q + v_q.$$

Then the insertion of these in (14) gives the reduced conditional equations

$$(14)' \qquad \begin{aligned} \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_q v_q + d_1 &= 0, \\ \beta_1 v_1 + \beta_2 v_2 + \ldots + \beta_q v_q + d_2 &= 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots& \\ \lambda_1 v_1 + \lambda_2 v_2 + \ldots + \lambda_q v_q + d_{n'} &= 0. \end{aligned}$$

For the sake of shortness, let these n' conditions be expressed by the notation $f(\alpha), f(\beta) \ldots f(\lambda)$.

Let $p_1, p_2 \ldots p_q$ be the weights of the observations $M_1, M_2 \ldots M_q$. The corrections $v_1, v_2 \ldots v_q$ are the same as the residual errors, and the sum of their weighted squares is represented by Σpv^2. The most probable values of $z_1, z_2 \ldots z_q$ are those that render a minimum the expression

$$\Sigma pv^2 - 2K_1 f(\alpha) - 2K_2 f(\beta) - \ldots - 2K_{n'} f(\lambda),$$

where $K_1, K_2 \ldots K_{n'}$ are multipliers, or "correlatives," of the conditional equations.*

The derivative of this expression with reference to each v is to be put separately equal to zero, thus :—

$$\begin{aligned} p_1 v_1 - (\alpha_1 K_1 + \beta_1 K_2 + \ldots + \lambda_1 K_{n'}) &= 0, \\ p_2 v_2 - (\alpha_2 K_1 + \beta_2 K_2 + \ldots + \lambda_2 K_{n'}) &= 0, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad & \\ p_q v_q - (\alpha_q K_1 + \beta_q K_2 + \ldots + \lambda_q K_{n'}) &= 0. \end{aligned}$$

These q equations together with the n' conditional equations are sufficient for the determination of the q residuals and the n' correlatives. The residuals may be written

$$(15) \qquad \begin{aligned} v_1 &= \frac{\alpha_1}{p_1} K_1 + \frac{\beta_1}{p_1} K_2 + \ldots + \frac{\lambda_1}{p_1} K_{n'}, \\ v_2 &= \frac{\alpha_2}{p_2} K_1 + \frac{\beta_2}{p_2} K_2 + \ldots + \frac{\lambda_2}{p_2} K_{n'}, \\ & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ v_q &= \frac{\alpha_q}{p_q} K_1 + \frac{\beta_q}{p_q} K_2 + \ldots + \frac{\lambda_q}{p_q} K_{n'}, \end{aligned}$$

* It is shown in works on the differential calculus, that the maximum or minimum of a function, $F(x, y, z)$, whose variables are connected by conditional equations, $\phi(x, y, z) = 0$, $\theta(x, y, z) = 0$, is to be found by multiplying the conditional equations by undetermined co-efficients, adding them to the function, and then,

and, if these be substituted in the reduced conditional equations, they become

$$
(16)\quad
\begin{aligned}
&\left[\frac{\alpha\alpha}{p}\right]K_1 + \left[\frac{\alpha\beta}{p}\right]K_2 + \ldots + \left[\frac{\alpha\lambda}{p}\right]K_{n'} + d_1 = 0,\\
&\left[\frac{\alpha\beta}{p}\right]K_1 + \left[\frac{\beta\beta}{p}\right]K_2 + \ldots + \left[\frac{\beta\lambda}{p}\right]K_{n'} + d_2 = 0,\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
&\left[\frac{\alpha\lambda}{p}\right]K_1 + \left[\frac{\beta\lambda}{p}\right]K_2 + \ldots + \left[\frac{\lambda\lambda}{p}\right]K_{n'} + d_{n'} = 0,
\end{aligned}
$$

in which the usual notation for sums is followed, for example:

$$\left[\frac{\alpha\beta}{p}\right] = \frac{\alpha_1\beta_1}{p_1} + \frac{\alpha_2\beta_2}{p_2} + \ldots + \frac{\alpha_q\beta_q}{p_q}.$$

The co-efficients in these equations have similar properties to those in the normal equations derived for independent observations, those of the first row being the same as those of the first column, and so on. Being n' in number they determine the n' correlatives; and the residuals v are then known. These residuals, applied as corrections to the observations, give the most probable values of the quantities $z_1, z_2 \ldots z_q$, and these must exactly satisfy the q conditional equations.

58*a*. As an illustrative example let there be five quantities connected by the two conditional equations,

$$z_1 + z_2 - z_3 = 0, \qquad z_2 - z_4 + z_5 = 0,$$

and let the results of five observations be

$$z_1 = 10.1, \quad z_2 = 6.6, \quad z_3 = 18.0, \quad z_4 = 9.2, \quad z_5 = 2.7 \text{ inches},$$

---

by the usual rule, determining the maximum or minimum of the new function, $F(x, y, z) + c\phi(x, y, z) + c'\theta(x, y, z)$.

The minimum Σpv^2 only gives the most probable values of the unknown quantities when these are independent. (See Art. 42.)

the weights of these observations being

$$p_1 = 1, \quad p_2 = 2, \quad p_3 = 1, \quad p_4 = 1, \quad p_5 = 1.$$

It is required to adjust the observations.

By comparing the conditional equations with (14) are found

$$\alpha_0 = 0, \quad \alpha_1 = +1, \quad \alpha_2 = +1, \quad \alpha_3 = -1, \quad \alpha_4 = 0, \quad \alpha_5 = 0,$$
$$\beta_0 = 0, \quad \beta_1 = 0, \quad \beta_2 = +1, \quad \beta_3 = 0, \quad \beta_4 = -1, \quad \beta_5 = +1.$$

Also by substituting the observed values in the conditional equations are derived $d_1 = -1.3$ and $d_2 = +0.1$. The coefficients of the equations (16) are next found; for example

$$\left[\frac{\alpha\alpha}{p}\right] = \frac{1}{1} + \frac{1}{2} + \frac{1}{1} + \frac{0}{1} + \frac{0}{1} = +2.5.$$

The correlative normal equations themselves then are

$$2.5K_1 + 0.5K_2 - 1.3 = 0,$$
$$0.5K_1 + 2.5K_2 + 0.1 = 0,$$

whence $K_1 = +0.55$ and $K_2 = -0.15$. From (15) the most probable corrections to the observed values are found to be

$$v_1 = +0.55, \quad v_2 = +0.20, \quad v_3 = -0.55, \quad v_4 = +0.15, \quad v_5 = -0.15,$$

and the final adjusted values are

$$z_1 = 10.65, \quad z_2 = 6.80, \quad z_3 = 17.45, \quad z_4 = 9.35, \quad z_5 = 2.55,$$

which exactly satisfy the two conditional equations.

This problem may be reduced to one of independent observations by eliminating two of the unknowns by means of the conditional equations. Thus, if z_3 and z_5 be eliminated the observation equations are $z_1 = 10.1$, $z_1 + z_2 = 18.0$, $z_4 = 9.2$, $-z + z_4 = 2.7$, all with weight 1, and $z_2 = 6.6$ with weight 2.

58*b*. A valuable check upon the solution of the correlative normal equations is given by the necessary relation,

$$\Sigma pv^2 + [Kd] = 0,$$

which may be proved in the following manner:

Let the first equation in (14)′ be multiplied by K_1, the second by K_2, the last by $K_{n'}$, and let the results be added, giving

$$\begin{aligned} &(\alpha_1 K_1 + \beta_1 K_2 + \ldots + \lambda_1 K_{n'})v_1 + K_1 d_1 \\ + &(\alpha_2 K_1 + \beta_2 K_2 + \ldots + \lambda_2 K_{n'})v_2 + K_2 d_2 \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ + &(\alpha_q K_1 + \beta_q K_2 + \ldots + \lambda_q K_{n'})v_q + K_{n'} d_{n'} = 0, \end{aligned}$$

Now, as shown on page 61, the coefficients of $v_1, v_2, \ldots v_q$, in this equation are $p_1 v_1, p_2 v_2, \ldots p_q v_q$. Hence

$$(p_1 v_1^2 + p_2 v_2^2 + \ldots + p_q v_q^2) + (K_1 d_1 + K_2 d_2 + \ldots + K_{n'} d_{n'}) = 0,$$

which may be abbreviated as is done above. Thus, if the residuals be computed from the observation equations, this relation furnishes a check on the numerical work.

For instance, in the example of the preceding article, the values of the residuals v have been found. Then

$$\Sigma pv^2 = 0.3025 + 0.0800 + 0.3025 + 0.0225 + 0.0225 = +0.73.$$

Further, the values of K and d are

$$K_1 = +0.55, \qquad K_2 = -0.15,$$
$$d_1 = -1.3, \qquad d_2 = +0.1,$$

and accordingly

$$[Kd] = +0.55(-1.3) - 0.15(+0.1) = -0.73.$$

The necessary relation $\Sigma pv^2 + [Kd] = 0$ is hence exactly satisfied, and the numerical work may be regarded as correct.

59. *Problems.*

1. Six indirect observations upon two quantities furnish the following observation equations:

$$
\begin{aligned}
+z_1 \qquad &= +3.01, \\
+2z_2 &= -1.20, \\
-z_1 + 3z_2 &= -4.65, \\
+2z_1 - z_2 &= +6.51, \\
z_1 + z_2 &= +2.35, \\
z_1 - z_2 &= +3.70.
\end{aligned}
$$

Form the two normal equations and find the most probable values of z_1 and z_2.

2. The bearing of a line is taken five times with a solar compass, giving the values

$$N.\ 12'\ E., \quad N.\ 7'\ E., \quad N.\ 10'\ W., \quad N.\ 2'\ W., \quad N.\ 2'.9\ E.$$

What is the adjusted bearing of the line if the weight of the last observation is five times that of each of the others?

3. Solve the following normal equations:

$$
\begin{aligned}
2z_1 - z_2 \qquad\qquad\qquad\qquad &+ 0.52 = 0, \\
-z_1 + 4z_2 - z_3 - z_4 \qquad\quad &- 0.26 = 0, \\
-z_2 + 2z_3 - z_4 \qquad\quad &+ 0.47 = 0, \\
-z_2 - z_3 + 3z_4 - z_5 &- 1.08 = 0, \\
-z_4 + 3z_5 &+ 0.34 = 0.
\end{aligned}
$$

4. A plane triangle has the angle A measured ten times, B measured five times, and C measured once. The sum of the three observed values is found to differ d seconds from 180 degrees. How shall this d be divided among the three angles?

CHAPTER IV.

THE PRECISION OF OBSERVATIONS.

60. In the adjustment of observations, it is often necessary to combine measurements of different degrees of precision; and for that purpose the determination of their weights is necessary. When the most probable or adjusted values have been obtained, it is also well to know what degree of confidence may be placed in them, so that comparisons may be made with values obtained under other circumstances. The comparison of observations is a very important part of the Method of Least Squares, since the knowledge of the value and precision of measurements is required for their most advantageous use. Moreover, the study of the precision of measurements is always necessary to improve and perfect the methods of observation.

The Probable Error.

61 The quantity usually selected to compare the precision of observations is the probable error, of which the following is a definition:

In any series of errors the probable error has such a value that the number of errors greater than it is the same as the number less than it. Or, it is an even wager that an error taken at random will be greater or less than the probable error.

The probable error is, then, the value of x in the probability integral (4) when $P = \frac{1}{2}$, or it is the value of x given by the equation

$$\frac{1}{2} = \frac{2}{\sqrt{\pi}} \int_0^{hx} e^{-h^2x^2} d.hx.$$

By interpolation from Table I, Chap. X, it is found that

$$P = 0.5 \quad \text{when} \quad hx = 0.4769.$$

Hence, denoting this value of x by r, the equation

$$(17) \qquad hr = 0.4769$$

gives the relation between the measure of precision h and the probable error r, and shows that h varies inversely as r.

62. To render more definite the conception of the measure of precision h and the probable error r, consider the case of two sets of observations made with different degrees of accuracy. Let the measure of precision of the first be h_1, and of the second h_2; then, from equation (2), the probability of errors in the first set will be represented by a curve whose equation is

$$y = h_1.dx.\pi^{-\frac{1}{2}} e^{-h_1^2x^2},$$

and for the second set by a curve

$$y = h_2.dx.\pi^{-\frac{1}{2}} e^{-h_2^2x^2},$$

in which dx is the constant difference between two consecutive errors. Now, suppose that the second set is twice as precise as the first, so that $h_1 = h$, and $h_2 = 2h$; then the equations will be

$$y = ahe^{-h^2x^2} \quad \text{and} \quad y = 2ahe^{-4h^2x^2},$$

in which a represents the constant $\pi^{-\frac{1}{2}}dx$. The curves corre-

sponding to these equations are given in Fig. 6; $XB_1A_1B_1X$ being the one for the set of observations whose measure of precision is h_1 or h, and $XB_2A_2B_2X$ the one for the set whose measure of precision is h_2, or $2h$. These curves show at a glance the relative probabilities of corresponding errors in the

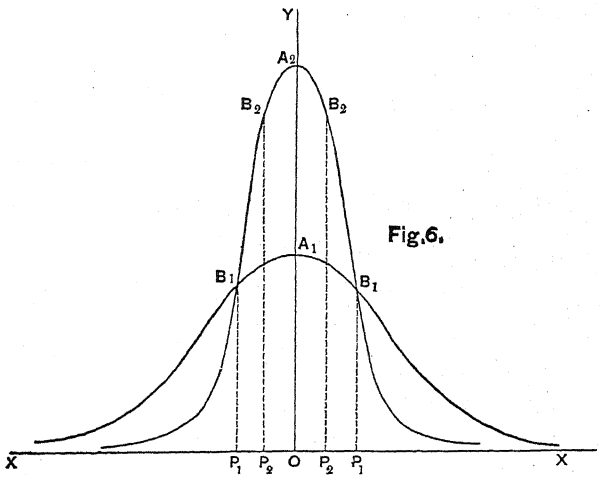


Fig. 6.

two sets: thus the probability of the error 0 is twice as much in the second as in the first set; the probability of the error OP_1 is nearly the same in each; while the probability of an error twice as large as OP_1 is much smaller in the second than in the first set. Now, if the lines P_1B_1, P_2B_2 be drawn so that the areas $P_1B_1A_1B_1P_1$ and $P_2B_2A_2B_2P_2$ are respectively one-half of the total areas of their corresponding curves, the line OP_1 will be the probable error of an observation in the first set, and OP_2 the probable error of one in the second set. Representing these by the letters r_1 and r_2, there must be in each case the constant relation

$$h_1r_1 = 0.4769, \quad h_2r_2 = 0.4769;$$

and, since h_2 is twice h_1, it follows that r_2 must be one-half of r_1.

The probable error, then, serves to compare the precision of observations equally as well as measures of precision. The smaller the probable error, the more precise are the measurements. For instance, if two sets of observations give for the length of a line in centimeters

$$L_1 = 427.32 \pm 0.04 \quad \text{and} \quad L_2 = 427.30 \pm 0.16,$$

in which 0.04 and 0.16 are the respective probable errors, the meaning is, that it is an even wager that the first is within 0.04 of the truth, and also an even wager that the second is within 0.16 of the true value; and the precision of the first result is to be regarded four times that of the second. The probable error thus serves as a means of comparison, and also gives an absolute idea of the uncertainty of the result.

63. In Art. 43 it was shown that the squares of measures of precision are directly proportional to weights; and in Art. 61 it is established that measures of precision are inversely proportional to probable errors. Hence the important relation:

Weights of observations are inversely proportional to the squares of their probable errors; or, in algebraic language,

$$(18) \qquad p_1 : p_2 : p :: \frac{1}{r_1^2} : \frac{1}{r_2^2} : \frac{1}{r^2}.$$

Weights and probable errors are constantly employed in the practical applications of the Method of Least Squares, while h is only needed in theoretic discussions. By means of the relation just established, the weights of observed results of different degrees of precision may be found from their computed

probable errors, and the observations be thus prepared for adjustment. For instance, in the two results

$$L_1 = 427.32 \pm 0.04, \quad L_2 = 427.30 \pm 0.16,$$

it is seen that the weight of 427.32 is sixteen times that of 427.30.

Probable Error of the Arithmetical Mean.

64. Let $M_1, M_2 \ldots M_n$ be n direct observations on the same quantity. The weight of each is 1, and the weight of their arithmetical mean is n. Let r be the probable error of a single observation, and r_0 the probable error of the arithmetical mean. The principle (18) of the last article gives

$$n : 1 :: \frac{1}{r_0^2} : \frac{1}{r^2},$$

from which

$$r_0 = \frac{r}{\sqrt{n}}; \tag{19}$$

or, the probable error of the arithmetical mean is equal to the probable error of a single observation divided by the square root of the number of observations.

The probable error of the mean, hence, decreases as $\sqrt{n}$ increases. If ten observations give a certain probable error for the mean, forty observations will be necessary in order to reduce it to one-half that value.

65. To find r, the probable error of a single observation, consider the fundamental law of the probability of error (2), or

$$y = h.dx.\pi^{-\frac{1}{2}}e^{-h^2x^2}.$$

By Art. 12 the probability of the occurrence of the independent errors $x_1, x_2 \ldots x_n$ is the product of the separate probabilities, or

$$P' = h^n (dx)^n \pi^{-\frac{n}{2}} e^{-h^2 \Sigma x^2}.$$

Now, for a given system of errors, it must be considered that the observations have been as precise as possible, or that h has such a value as to render P' a maximum. Putting the first derivative $\frac{dP'}{dh}$ equal to zero, and reducing, gives

$$n - 2h^2 \Sigma x^2 = 0, \quad \text{or} \quad h = \sqrt{\frac{n}{2\Sigma x^2}}.$$

Since, by Art. 61, hr equals the constant 0.4769,

$$r = \frac{0.4769}{h} = 0.6745 \sqrt{\frac{\Sigma x^2}{n}}.$$

Here Σx^2 is the sum of the squares of the true errors, which are unknown. In a large number of observations the errors closely agree with the residuals, and Σx^2 may be taken as equal to Σv^2; but, for a limited number of errors, Σv^2 is less than Σx^2, since, by the principle of Least Squares, the first is the minimum value of the second; so that

$$\Sigma x^2 = \Sigma v^2 + u^2,$$

where u^2 is a quantity as yet undetermined. The absolute value of u^2 cannot be found; but it is known to decrease as n increases, and for a given number of residuals to increase when Σx^2 increases: as the best approximation, u^2 may be taken as equal to $\frac{\Sigma x^2}{n}$. Then

$$\Sigma x^2 = \Sigma v^2 + \frac{\Sigma x^2}{n}, \quad \text{or} \quad \frac{\Sigma x^2}{n} = \frac{\Sigma v^2}{n-1};$$

and, inserting this in the above value of r, it becomes

$$(20) \qquad r = 0.6745\sqrt{\frac{\Sigma v^2}{n-1}}.$$

This is the formula for the probable error of a single direct observation, or of an observation of the weight unity. To use it, the residuals are to be found by subtracting each measurement from the arithmetical mean, and the sum of their squares then formed. When r is known, the probable error r_0 of the arithmetical mean is found by the formula (19), or it may be written at once

$$(21) \qquad r_0 = 0.6745\sqrt{\frac{\Sigma v^2}{n(n-1)}},$$

which is the usual form for computation.

Probable Error of the General Mean.

66. Let $M_1, M_2 \ldots M_n$ be n direct observations having the weights $p_1, p_2 \ldots p_n$. The weight of the general mean is $p_1 + p_2 + \ldots + p_n$, or Σp. Let r be the probable error of an observation of the weight unity, and r_0 the probable error of the mean. Then, from the fundamental relation between weights and probable errors,

$$1 : \Sigma p :: \frac{1}{r^2} : \frac{1}{r_0^2},$$

from which the probable error of the mean is

$$(22) \qquad r_0 = \frac{r}{\sqrt{\Sigma p}};$$

and, in general, the probable error of any observation is equal to r divided by the square root of its weight. To find r, an

investigation like that in the preceding article could be employed; but it may be well to give one of a different character.

67. Let h be the measure of precision of an observation of the weight unity, and $h_1, h_2 \dots h_n$ those of the observations whose weights are $p_1, p_2 \dots p_n$. By formula (7) the quantities $h_1, h_2 \dots h_n$ may be expressed in terms of the weights, thus:

$$h_1^2 = p_1 h^2, \quad h_2^2 = p_2 h^2 \dots h_n^2 = p_n h^2;$$

and, in general, if x be any error, p the weight of the corresponding observation, and h the measure of precision of an observation of the weight unity, the probability y is, from (2),

$$y = hp^{\frac{1}{2}}\pi^{-\frac{1}{2}}dx.e^{-h^2px^2}.$$

Now, the quantity Σpx^2y is the same as $\dfrac{\Sigma px^2}{n}$, since each term, such as $p_2x_2^2$, occurs ny_2 times in n observations; and, for a continuous series of errors,

$$\frac{\Sigma px^2}{n} = \frac{h\sqrt{p}}{\sqrt{\pi}}\int_{-\infty}^{+\infty} px^2 e^{-h^2px^2}dx.$$

Taking in this $hx\sqrt{p} = t$ as the unit variable, it may be written

$$\frac{\Sigma px^2}{n} = \frac{1}{h^2\sqrt{\pi}}\int_{-\infty}^{+\infty} t^2 e^{-t^2}dt.$$

The value of the integral in this expression is $\dfrac{\sqrt{\pi}}{2}$,* and hence

$$\frac{\Sigma px^2}{n} = \frac{1}{2h^2}.$$

---

* From the footnote to Art. 31,

$$\int_0^\infty e^{-t^2}dt = \frac{\sqrt{\pi}}{2}.$$

From Art. 61 the value of $\frac{1}{h^2}$ is $\left(\frac{r}{0.4769}\right)^2$: hence

$$r = 0.6745 \sqrt{\frac{\Sigma px^2}{n}}$$

is the probable error of an observation of the weight unity.

Now, Σpx^2 is in terms of the true unknown errors, and is greater than Σpv^2. Place, then,

$$\Sigma px^2 = \Sigma pv^2 + u^2,$$

in which u^2 is a quantity to be determined. The probability, P', of the system of errors, is

$$P' = Ke^{-h^2\Sigma px^2} = Ke^{-h^2(\Sigma pv^2 + u^2)} = K'e^{-h^2u^2}.$$

Here it is seen that the law of probability of u^2 is similar to that of an error x; and, as in Art. 31, it may be shown that the constant K' is $h.\pi^{-\frac{1}{2}}du$. The mean of all the possible values of u^2 is, then,

$$\frac{h}{\sqrt{\pi}}\int_{-\infty}^{+\infty} u^2e^{-h^2u^2}du = \frac{1}{2h^2},$$

Placing $t = t\sqrt{s}$, this becomes

$$\int_0^\infty e^{-t^2s}dt = \frac{\sqrt{\pi}}{2\sqrt{s}}.$$

Differentiating this equation with reference to s, and regarding t as constant.

$$-\int_0^\infty e^{-t^2s}t^2dsdt = -\frac{\sqrt{\pi}ds}{4\sqrt{s^3}}.$$

Dividing this by $-ds$, and making $s = 1$, it becomes

$$\int_0^\infty e^{-t^2}t^2dt = \frac{\sqrt{\pi}}{4} = \text{one-half of the integral above.}$$

and this must be taken as the best attainable value of u^2. But it was shown that $\frac{1}{2h^2}$ is equal to $\frac{\Sigma px^2}{n}$. Hence

$$\Sigma px^2 = \Sigma pv^2 + \frac{\Sigma px^2}{n},$$

from which

$$\frac{\Sigma px^2}{n} = \frac{\Sigma pv^2}{n - 1};$$

and therefore the probable error r becomes

$$(23) \qquad r = 0.6745 \sqrt{\frac{\Sigma pv^2}{n - 1}}.$$

The probable error of the general mean is now, from **(22)**,

$$(24) \qquad r_0 = 0.6745 \sqrt{\frac{\Sigma pv^2}{(n - 1)\Sigma p}}.$$

If the observations be all of the weight unity, Σp becomes n, and the formulas (23) and (24) agree with (20) and (21). The probable error of any observation whose weight is p is found by dividing r by the square root of p.

Laws of Propagation of Error.

68. Let z_1 and z_2 be two independently measured quantities whose probable errors are r_1 and r_2. It is required to find the probable error R of the sum $z_1 + z_2$, or of the difference $z_1 - z_2$. Let $Z = z_1 \pm z_2$, and let the errors arising in the two cases be,

$$x_1', \; x_1'', \; x_1''', \text{ etc., for } z_1,$$
$$x_2', \; x_2'', \; x_2''', \text{ etc., for } z_2.$$

Then the corresponding errors of Z are

$$X' = x_1' \pm x_2', \quad X'' = x_1'' \pm x_2'', \quad X''' = x_1''' \pm x_2''', \text{ etc.}$$

Squaring each X, and adding the results, gives

$$\Sigma X^2 = \Sigma x_1^2 \pm 2\Sigma x_1 x_2 + \Sigma x_2^2.$$

The products $x_1 x_2$ will be both positive and negative, and, on the average, $\Sigma x_1 x_2 = 0$: hence

$$\Sigma X^2 = \Sigma x_1^2 + \Sigma x_2^2;$$

and, if n be the number of errors,

$$\frac{\Sigma X^2}{n} = \frac{\Sigma x_1^2}{n} + \frac{\Sigma x_2^2}{n}.$$

Now, by Art. 65, it is known that $\frac{\Sigma x^2}{n}$ varies with r^2: hence, for the case in hand,

$$(25) \qquad R^2 = r_1^2 + r_2^2;$$

from which the probable error of Z is known.

In like manner, if Z be the sum or difference of several independent quantities, namely, if

$$Z = z_1 \pm z_2 \pm z_3 \pm \ldots \pm z_m,$$

then the probable error of Z is given by the relation,

$$(26) \qquad R^2 = r_1^2 + r_2^2 + r_3^2 + \ldots + r_m^2.$$

This formula is very important in the discussion of linear measurements.

69. Secondly, consider Z to be a multiple of an observed quantity z, so that $Z = Az$, where A is a known number. Then an error x in z produces an error Ax in Z, and

$$X = Ax, \quad X^2 = A^2x^2, \text{ and } \Sigma X^2 = A^2\Sigma x^2.$$

Hence, as before, it is to be concluded that

$$(27) \qquad R^2 = A^2r^2, \text{ or } R = Ar.$$

By combining the principle of the last article with that just deduced, it is seen, if

$$Z = Az_1 \pm Bz_2 \pm Cz_3 \pm \text{etc.},$$

and if the probable errors of z_1, z_2, z_3 are r_1, r_2, r_3, that the probable error of Z is given by

$$(28) \qquad R^2 = A^2r_1^2 + B^2r_2^2 + C^2r_2^2 + \text{etc.},$$

which is a more general formula than (26).

It is interesting to note that formula (19) can be deduced from (26), and also (22) from (28). Thus, if z_1, $z_2 \ldots z_n$ are n observed values of the same quantity, the probable error of their sum is, by (26),

$$R = \sqrt{r_1^2 + r_2^2 + \ldots + r_n^2} = \sqrt{nr^2},$$

and by (27) the probable error of $\frac{1}{n}$th of this sum is

$$r_0 = \frac{\sqrt{nr^2}}{n} = \frac{r}{\sqrt{n}};$$

which is the probable error of the arithmetical mean, as in (19).

70. Next, consider Z to be the product of two independently observed quantities z_1 and z_2, whose probable errors are r_1 and r_2. Let X be an error in Z corresponding to the errors x_1 and x_2 in z_1 and z_2: then

$$Z + X = (z_1 + x_1)(z_2 + x_2) = z_1 z_2 + z_1 x_2 + z_2 x_1 + x_1 x_2.$$

Here $Z = z_1 z_2$, and $x_1 x_2$ vanishes in comparison with $z_1 x_2$ and $z_2 x_1$; so that

$$X = z_1 x_2 + z_2 x_1.$$

Squaring each error X, and taking the sums, gives

$$\Sigma X^2 = \Sigma z_1^2 x_2^2 + \Sigma z_2^2 x_1^2 + 2\Sigma z_1 z_2 x_1 x_2,$$

the last term of which vanishes, since the product $x_1 x_2$ is as likely to be positive as negative: hence

$$\Sigma X^2 = z_1^2 \Sigma x_2^2 + z_2^2 \Sigma x_1^2,$$

and accordingly, as in Art. 68,

$$R^2 = z_1^2 r_2^2 + z_2^2 r_1^2, \tag{29}$$

from which the probable error of Z may be computed.

71. Lastly, let Z be any function of the independently observed quantities $z_1, z_2, z_3 \ldots$, or $Z = f(z_1, z_2, z_3 \ldots)$, and let it be required to find the probable error R of Z from the probable errors $r_1, r_2, r_3 \ldots$ of the observed quantities. Take x_1, x_2, x_3 as any errors in z_1, z_2, z_3, and X as the corresponding error in Z: then

$$Z + X = f[(z_1 + x_1),\ (z_2 + x_2),\ (z_3 + x_3) \ldots].$$

Now, if these errors are so small, that their second and higher

powers may be neglected, the development of the function by Taylor's theorem gives

$$X = \frac{dZ}{dz_1}x_1 + \frac{dZ}{dz_2}x_2 + \frac{dZ}{dz_3}x_3 + \ldots$$

Accordingly, by the same reasoning as in the previous articles,

$$(30) \qquad R^2 = \left(\frac{dZ}{dz_1}\right)^2 r_1^2 + \left(\frac{dZ}{dz_2}\right)^2 r_2^2 + \left(\frac{dZ}{dz_3}\right)^2 r_3^2 + \ldots;$$

which is a general formula appplicable to all functions.

The laws of propagation of error, given by formulas (25) to (30), are very important in forming proper rules for taking observations, as well as in discussing and comparing results. The law $R = \sqrt{r_1^2 + r_2^2}$, which gives the probable error of Z when $Z = z_1 + z_2$, or when $Z = z_1 - z_2$, has been likened by Jordan to the celebrated geometrical theorem of Pythagoras.

Probable Errors for Independent Observations.

72. In Arts. 46–50 are given methods of finding the most probable values of independent quantities which are indirectly observed. To determine the probable errors of any adjusted value, z, let p_z denote its weight, and r_z its probable error. Then, if r be the probable error of an observation whose weight is unity, the relation (18) gives

$$p_z : 1 :: \frac{1}{r_z^2} : \frac{1}{r^2},$$

from which

$$(31) \qquad r_z = \frac{r}{\sqrt{p_z}}.$$

Hence, in order to find the probable errors of z_1, z_2 . . . z_q, it is

necessary to find r and their weights. And, in general, the probable error of any observation is equal to r divided by the square root of its weight.

73. To find the probable error of an observation whose weight is unity, the following reasoning may be employed:

Suppose that the normal equations (13) have been solved, and the most probable values $z_1, z_2 \ldots z_q$ deduced. Let the corresponding true values be represented by $z_1 + \delta z_1$, $z_2 + \delta z_2 \ldots z_q + \delta z_q$, in which $\delta z_1, \delta z_2 \ldots \delta z_q$ are small unknown corrections. Now if, in the observation equations (12), the most probable values be substituted, they will not reduce to zero, but leave small residuals $v_1, v_2 \ldots v_n$; thus:

$$\begin{aligned}
a_1 z_1 + b_1 z_2 + \ldots + l_1 z_q + m_1 &= v_1 \quad \text{with weight } p_1,\\
a_2 z_1 + b_2 z_2 + \ldots + l_2 z_q + m_1 &= v_2, \quad \text{with weight } p_2,\\
\cdots\cdots\cdots\cdots\cdots\\
a_n z_1 + b_n z_2 + \ldots + l_n z_q + m_n &= v_n, \quad \text{with weight } p_n,
\end{aligned}$$

while, if the corresponding true values be substituted, they will give the true errors; thus:

$$\begin{aligned}
a_1 (z_1 + \delta z_1) + b_1 (z_2 + \delta z_2) + \ldots + m_1 &= x_1,\\
a_2 (z_1 + \delta z_1) + b_2 (z_2 + \delta z_2) + \ldots + m_2 &= x_2,\\
\cdots\cdots\cdots\cdots\cdots\\
a_n (z_1 + \delta z_1) + b_n (z_2 + \delta z_2) + \ldots + m_3 &= x_n.
\end{aligned}$$

Subtracting each one of the former equations from the latter gives the following residual equations:

$$\begin{aligned}
v_1 + a_1 \delta z_1 + b_1 \delta z_2 + \ldots + l_1 \delta z_q &= x_1,\\
v_2 + a_2 \delta z_1 + b_2 \delta z_2 + \ldots + l_2 \delta z_q &= x_2,\\
\cdots\cdots\cdots\cdots\cdots\\
v_n + a_n \delta z_1 + b_n \delta z_2 + \ldots + l_n \delta z_q &= x_n.
\end{aligned}$$

Now, the principle of Least Squares (6) requires that Σpx^2 shall be made a minimum to give the most probable values of z_1,

$z_2 \ldots z_q$; and, by the solution of the normal equations, its minimum value is the sum Σpv^2. From the residual equations a relation connecting the two sums Σpv^2 and Σpx^2 may be found by squaring both members of each of those equations, multiplying each by its corresponding weight, and then adding the products. Without actually performing these operations, it is evident, that if the squares and products of $\delta z_1, \delta z_2 \ldots \delta z_q$ be neglected as small in comparison with the first powers, the result will be of the form

$$\Sigma pv^2 + k_1\delta z_1 + k_2\delta z_2 + \ldots + k_q\delta z_q = \Sigma px^2,$$

in which $k_1, k_2 \ldots k_q$ are co-efficients of the unknown corrections, and dependent only upon the known co-efficients and weights. If the number of unknown quantities is q, there will be q of these terms. Placing

$$k_1\delta z_1 = u_1^2, \quad k_2\delta z_2 = u_2^2 \ldots k_q\delta z_q = u_q^2,$$

it becomes

$$\Sigma pv^2 + u_1^2 + u_2^2 + \ldots + u_q^2 = \Sigma px^2.$$

Now, the probability of the occurrence of the error x_1, whose measure of precision is h_1, and whose weight is p_1, is, by (2) and (7),

$$y_1 = hp_1^{\frac{1}{2}}.dx.\pi^{-\frac{1}{2}}e^{-h^2p_1x_1^2},$$

in which h is the measure of precision of an observation of the weight 1. And hence, by exactly the same reasoning as in Art. 67, it may be shown, that, when n is a large number,

$$\Sigma px^2 = \frac{n}{2h^2}.$$

Further: if there be but one unknown quantity, there is but one u^2, whose value, as shown in Art. 67, is $\frac{1}{2h^2}$. And, since

this is true whichever unknown quantity be considered, the value of each u^2 must be $\frac{1}{2h^2}$; and, as there are q of these values, the above result becomes

$$\Sigma pv^2 + \frac{q}{2h^2} = \frac{n}{2h^2},$$

from which

$$h = \sqrt{\frac{n-q}{2\Sigma pv^2}}.$$

Therefore, from the constant relation (17) between h and r, the probable error of an observation of the weight unity is

$$(32) \qquad r = 0.6745 \sqrt{\frac{\Sigma pv^2}{n-q}}.$$

74. The probable errors of the values $z_1, z_2 \ldots z_q$ can now be found from (31) as soon as their weights are known. These will now be determined.

The observations $M_1, M_2 \ldots M_n$ furnish the observation equations (12) and the normal equations (13). The solution of the latter gives the values of $z_1, z_2 \ldots z_q$ in terms of M_1, $M_2 \ldots M_n$, and co-efficients independent of those quantities. Suppose the general solution to give

$$\begin{aligned} z_1 &= \sigma_1 M_1 + \sigma_2 M_2 + \sigma_3 M_3 + \ldots + \sigma_n M_n, \\ z_2 &= \tau_1 M_1 + \tau_2 M_2 + \tau_3 M_3 + \ldots + \tau_n M_n, \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ z_q &= \zeta_1 M_1 + \zeta_2 M_2 + \zeta_3 M_3 + \ldots + \zeta_n M_n, \end{aligned}$$

in which the co-efficients $\sigma, \tau \ldots \zeta$ depend only upon the constants $a, b \ldots l$ and the weights $p_1, p_2 \ldots p_n$. Then, if R_{z_1} is the probable error of z_1, and $r_1, r_2 \ldots r_n$ are the probable errors of $M_1, M_2 \ldots M_n$, the formula (28) gives

$$R_{z_1}^2 = \sigma_1^2 r_1^2 + \sigma_2^2 r_2^2 + \ldots + \sigma_n^2 r_n^2.$$

Now, since the squares of probable errors are inversely proportional to weights, this becomes

$$\frac{1}{p_{z_1}} = \frac{\sigma_1^2}{p_1} + \frac{\sigma_2^2}{p_2} + \ldots + \frac{\sigma_n^2}{p_n} = \left[\frac{\sigma\sigma}{p}\right],$$

from which the value of p_{z_1} is

$$p_{z_1} = \frac{1}{\left[\frac{\sigma\sigma}{p}\right]}.$$

In like manner it is easy to show that the weight of z_2 is the reciprocal of $\left[\frac{\tau\tau}{p}\right]$, and that the weight of z_q is the reciprocal of $\left[\frac{\zeta\zeta}{p}\right]$.

Owing, however, to the labor of finding the co-efficients $\sigma, \tau \ldots \zeta$, it is better to deduce these expressions under a different form. Suppose the normal equations to be solved, giving

$$\begin{aligned} z_1 &= \alpha_1[paM] + \alpha_2[pbM] + \ldots + \alpha_q[plM], \\ z_2 &= \beta_1[paM] + \beta_2[pbM] + \ldots + \beta_q[plM], \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ z_q &= \lambda_1[paM] + \lambda_2[pbM] + \ldots + \lambda_q[plM], \end{aligned}$$

in which $\alpha, \beta \ldots \lambda$ are co-efficients independent of $M_1, M_2 \ldots M_n$. Then the respective weights of $z_1, z_2 \ldots z_q$ will be $\frac{1}{\alpha_1}, \frac{1}{\beta_2} \ldots \frac{1}{\lambda_q}$. To show this, it will be sufficient to consider the quantity z_2, and to prove that $\left[\frac{\tau\tau}{p}\right] = \beta_2$. By comparing the above two expressions for z_2, it is seen that

$$\begin{aligned} \tau_1 &= \beta_1 p_1 a_1 + \beta_2 p_1 b_1 + \ldots + \beta_q p_1 l_1, \\ \tau_2 &= \beta_1 p_2 a_2 + \beta_2 p_2 b_2 + \ldots + \beta_q p_2 l_2, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ \tau_n &= \beta_1 p_n a_n + \beta_2 p_n b_n + \ldots + \beta_q p_n l_n. \end{aligned}$$

Squaring each of these equations, dividing each by its p, and adding the results, gives

$$\left[\frac{\tau\tau}{p}\right] = \beta_1(\beta_1[paa] + \beta_2[pab] + \ldots + \beta_q[pal])$$
$$+ \beta_2(\beta_1[pab] + \beta_2[pbb] + \ldots + \beta_q[pbl]) + \ldots$$
$$+ \beta_q(\beta_1[pal] + \beta_2[pbl] + \ldots + \beta_q[pll]).$$

Now, if the normal equations (13) are solved by the method of undetermined multipliers, the first is to be multiplied by a number β_1, the second by β_2, the q^{th} by β_q, and the products added. Then, if upon these multipliers the following conditions be imposed,

$$\begin{aligned}
\beta_1[paa] + \beta_2[pab] + \ldots + \beta_q[pal] &= 0,\\
\beta_1[pab] + \beta_2[pbb] + \ldots + \beta_q[pbl] &= 1,\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\beta_1[pal] + \beta_2[pbl] + \ldots + \beta_q[pll] &= 0,
\end{aligned}$$

all the terms except those involving β_2 will reduce to zero, and the value of β_2 will be the same as above expressed. Accordingly $\left[\frac{\tau\tau}{p}\right] = \beta_2$, and the weight of z_2 is $\frac{1}{\beta_2}$; which was to be proved.

75. The following is hence a method of finding the weights of the values of the unknown quantities. Preserve the absolute terms of the normal equations in literal form during the solution. Then the weight of any value, as z_3, is equal to the reciprocal of the co-efficient of the absolute term which belonged to the normal equation for z_3.

For example, take the normal equations

$$\begin{aligned}
3z_1 - z_2 - z_3 &= A_1,\\
-z_1 + 3z_2 - z_3 &= A_2,\\
-z_1 - z_2 + 2z_3 &= A_3.
\end{aligned}$$

The solution of these by any method gives

$$z_1 = \tfrac{5}{8}A_1 + \tfrac{3}{8}A_2 + \tfrac{1}{2}A_3,$$
$$z_2 = \tfrac{3}{8}A_1 + \tfrac{5}{8}A_2 + \tfrac{1}{2}A_3,$$
$$z_3 = \tfrac{1}{2}A_1 + \tfrac{1}{2}A_2 + \;\; A_3,$$

and hence the weight of z_1 is $\frac{8}{5}$, the weight of z_2 is $\frac{8}{5}$, and the weight of z_3 is 1. It is evident, if it be only desired to find the weight of z_1, that A_2 and A_3 need not be retained in the computation, but may be made zero. So, in finding the weight of z_2, only A_2 need be retained in the work.

76. As an illustration of the preceding principles, let there be three observation equations of weight unity,

$$z_1 = 0, \quad z_2 = 0, \quad z_1 - z_2 = +0.51.$$

The normal equations are

$$2z_1 - z_2 = +0.51, \quad -z_1 + 2z_2 = -0.51.$$

Writing A_1 and A_2 for the absolute terms the solution of these equations gives

$$z_1 = \frac{2}{3}A_1 + \frac{1}{3}A_2, \qquad z_2 = \frac{1}{3}A_1 + \frac{2}{3}A_2,$$

from which the adjusted probable values are $z_1 = +0.17$ and $z_2 = -0.17$, while the weight of each of these values is seen to be $1\frac{1}{2}$. The sum of the squares of the residuals is $\Sigma v^2 = 0.0578$, and from (32) the probable error of an observation of weight unity is ± 0.16. This divided by $\sqrt{1.5}$ gives ± 0.13 as the probable error of the adjusted values of z_1 and z_2. The adjusted value of the third observation is $z_1 - z_2 = +0.34$, and by (25) the probable error of this value is ± 0.18. It is seen that the corrections to the three observed values are here numerically equal.

Probable Errors for Conditioned Observations.

77. When conditioned observations are adjusted by the general method of Art. 57, where the q unknown quantities in the n observation equations are reduced to $q - n'$ independent quantities by means of the n' conditional equations, the probable error of an observation of the weight unity is evidently given by the formula (32), if q be replaced by $q - n'$, or

$$(33) \qquad r = 0.6745\sqrt{\frac{\Sigma pv^2}{n - q + n'}};$$

and the probable errors of observations or values whose weights are p_1, p_2, etc., are, by (31),

$$r_1 = \frac{r}{\sqrt{p_1}}, \quad r_2 = \frac{r}{\sqrt{p_2}}, \text{ etc.}$$

The weights of $z_1, z_2 \ldots z_q$ are to be found exactly as in Art. 75.

For the case of direct observations on several quantities adjusted by the method of Art. 58, the number of observation equations is the same as that of the unknown quantities, or $n = q$; and, if n' be the number of conditional equations, the probable error of an observation of the weight unity is

$$(34) \qquad r = 0.6745\sqrt{\frac{\Sigma pv^2}{n'}},$$

from which the probable error of any observation of given weight can at once be deduced. In this case the residuals v are merely the differences between the observed and the adjusted values.

78. *Problems.*

1. There are two series of observations of an angle, each taken to hundredths of a second. The probable error of a single observation in the first series is $0''.65$, and in the second $1''.45$. Compute the probabilities of the error $0''.00$ and of the error $2''.00$ in the two cases.

2. It is required to determine the value of an angle with a probable error of $0''.25$. Twenty measurements give a mean whose probable error is $0''.38$. How many additional measurements are necessary?

3. Find the probable error of the mean of two observations which differ by the amount a.

4. Let z_1, z_2, and z_3 be independently observed quantities whose probable errors are r_1, r_2, and r_3. If $Z = z_1^2 + z_2^2 + z_3^2$ find the probable error of Z.

5. Let r be the probable error in $\log a$. What is the probable error in the number a?

6. Given the following observation equations:—

$$\begin{aligned} z_1 &= 4.5, &&\text{with weight } 10,\\ z_2 &= 1.6, &&\text{with weight } 5,\\ z_1 - z_2 &= 2.7, &&\text{with weight } 3. \end{aligned}$$

What are the most probable values of z_1 and z_2 with their probable errors?

7. Given the observation equations (all of equal weight)

$$\begin{aligned} 2z_1 - z_2 + z_3 &= 3,\\ 3z_1 + 3z_2 - z_3 &= 14,\\ 4z_1 + z_2 + 4z_3 &= 21,\\ -5z_1 + 2z_2 + 3z_3 &= 5, \end{aligned}$$

to find the best values of z_1, z_2, and z_3, with their probable errors.

CHAPTER V.

DIRECT OBSERVATIONS ON A SINGLE QUANTITY.

79. In the preceding pages the fundamental methods and formulas for the adjustment and comparison of observations have been deduced. In this and the three following chapters the application of these methods to practical examples will be presented. The most common case of observation is that of direct measurements on a single quantity, and this will form the subject of the present chapter.

Observations of Equal Weight.

80. When a quantity is measured several times with equal care, so that there is no reason for preferring one observation to another, the observations are of equal weight. From remote antiquity the arithmetical mean of the measurements has always been regarded as the best or most probable value of the quantity sought; and, as shown in Art. 44, this is confirmed by the fundamental principle of the Method of Least Squares.

Let z be the most probable value of the measured quantity, n the number of observations, and M any observation. Let r be the probable error of a single observation, and r_0 the proba-

ble error of the adjusted value z. Let also v be any residual obtained by subtracting M from z.

The most probable value of the quantity is the arithmetical mean, expressed, as in Art. 44, by formula (8),

$$z = \frac{\Sigma M}{n}.$$

The probable error of a single observation, as shown in Art. 65, is, by formula (20),

$$r = 0.6745 \sqrt{\frac{\Sigma v^2}{n - 1}}.$$

Lastly, as shown in Art. 64, the probable error of the mean is, by (19),

$$r_0 = \frac{r}{\sqrt{n}}.$$

Formula (8) indicates the method of adjustment, while (20) and (19) determine the precision of observation and of the mean. After finding z, each observation is subtracted from it, giving n values of v. The squares of these are taken, and their sum is Σv^2; then r is computed, and lastly, r_0. If desired, r_0 can be also found directly from formula (21),

$$r_0 = 0.6745 \sqrt{\frac{\Sigma v^2}{n(n - 1)}},$$

which is the same as (19).

81. As an example, consider the following twenty-four measurements of an angle of the primary triangulation of the United-States Coast-Survey, made at the station Pocasset in Massachusetts, and recorded in the Report for 1854:

Observations.	v.	v^2.
116°43′44″.45	5.19	26.94
50.55	−0.91	.83
50.95	−1.31	1.72
48.90	0.74	.55
49.20	0.44	.19
48.85	0.79	.63
47.40	2.24	5.02
47.75	1.89	3.57
51.05	−1.41	2.00
47.85	1.79	3.20
50.60	−0.96	.92
48.45	1.19	1.42
51.75	−2.11	4.45
49.00	0.64	.41
52.35	−2.71	7.34
51.30	−1.66	2.75
51.05	−1.41	2.00
51.70	−2.06	4.24
49.05	0.59	.35
50.55	−0.91	.83
49.25	0.39	.15
46.75	2.89	8.35
49.25	0.39	.15
53.40	−3.76	14.14
z = 116°43′49″.64	$\Sigma v^2 = 92.15$	

The most probable value of the angle is found by adding the observations, and dividing the sum by twenty-four. This is 116° 43′ 49″.64. Subtracting from this the first reading gives

5.19 for the first residual, which is placed in the column headed v. The square of this is 26.94, which is placed in the column headed v^2. The sum of all these squares is 92.15. Then from (20) the probable error of a single observation is

$$r = 0.6745\sqrt{\frac{92.15}{23}} = 1''.35\,;$$

and the probable error of the mean is, from (19),

$$r_0 = \frac{1.35}{\sqrt{24}} = 0''.28:$$

hence the final value may be written $116^\circ\ 43'\ 49''.64 \pm 0''.28$.

The precision of the mean of these twenty-four observations is such that $0''.28$ is to be regarded as the error to which it is liable; that is, it is an even wager that the mean differs from the true value of the angle by less than $0''.28$, and of course also an even wager that it differs by more than $0''.28$. The precision of a single observation is such that $1''.35$ is the error to which it is liable; that is, half the errors should be less, and half greater, than $1''.35$ in a large number of observations. It will be noticed that twelve of the above residuals are less, and twelve greater, than $1''.35$.

In Art. 27 it was shown that the algebraic sum of the residuals must always equal zero. This principle may be used to furnish a check on the accuracy of the numerical work.

82. The tables in Chap. X will be found useful in abbreviating computations. By the help of Table VI the squares of the residuals can be readily found. By Table III the computation of r and r_0 can be much abridged; for instance, in the case of the last article, $n = 24$, and

$$r = 0.1406\sqrt{92.15} = 1''.35,$$
$$r_0 = 0.0287\sqrt{92.15} = 0\ .28.$$

The table of four-figure logarithms will also prove useful in extracting roots and performing multiplications.

When the tables are used, it will be found more convenient to compute r_0 from (21) than from (19). Formula (19), however, is very important in indicating that the probable error of the mean decreases, and hence that its precision increases, with the square root of the number of observations.

It should be borne in mind, that the method of the arithmetical mean only applies to equally good observations on a single quantity, and that it cannot be used for the adjustment of observations on several related quantities. For instance, let an angle be measured, and found to be $60\frac{1}{2}$ degrees, and again let it be measured in two parts, one being found to be 40 degrees, and the other 20 degrees. The proper adjusted value of the angle is not, as might at first be supposed, the mean of $60\frac{1}{2}$ and 60, which is $60\frac{1}{4}$ degrees, but, as will be seen in the next chapter, it is $60\frac{1}{3}$ degrees,—a result which requires the correction of each observation by the same amount.

Shorter Formulas for Probable Error.

83. The method of computing probable errors by formula (20) is that considered the best by all writers. Nevertheless, on account of the labor of forming the squares of the residuals, a simpler and less accurate formula is often employed, in which only the residuals themselves are used. To deduce it, let n be the number of observations, and Σv the sum of the residuals, all taken with the positive sign, and Σx the sum of all the errors taken positively. Then $\frac{\Sigma x}{n}$ is the mean of the errors; and, by the same reasoning as in Art. 67, this mean is

$$\frac{\Sigma x}{n} = \frac{2h}{\sqrt{\pi}}\int_0^\infty x e^{-h^2x^2}dx = \frac{1}{h\sqrt{\pi}}.$$

Now since, by Art. 61, the product hr is equal to the constant 0.4769, the value of r in terms of Σx is

$$r = 0.8453 \frac{\Sigma x}{n}.$$

The sum of the errors Σx is in general different from the sum of the residuals Σv. Both in Art. 65 and Art. 67 it was shown that

$$\frac{\Sigma x^2}{n} = \frac{\Sigma v^2}{n - 1};$$

and it may hence be concluded, that, on the average, x^2 is greater than v^2 in the ratio of n to $n - 1$, and that, on the average, x is greater than v in the ratio of $\sqrt{n}$ to $\sqrt{n - 1}$, or that

$$\frac{\Sigma x}{\sqrt{n}} = \frac{\Sigma v}{\sqrt{n - 1}}.$$

Accordingly the above value of r becomes

$$r = \frac{0.8453 \Sigma v}{\sqrt{n(n - 1)}}, \tag{35}$$

which gives the probable error of a single observation. By substituting this in (19), the value of r_0 becomes

$$r_0 = \frac{0.8453 \Sigma v}{n\sqrt{n - 1}}, \tag{36}$$

which is the probable error of the arithmetical mean.

84. Formulas (35) and (36) will be found much easier to use than (20) and (21). In Table IV the co-efficients of Σv are tabulated for values of n from 2 to 100, and by its use the computations are much abridged.

As an example, consider the following eight measurements of a line made with a tape twenty meters long, graduated to centimeters:

Observations.	v.
188.97	0.095
.88	.005
.91	.035
.99	.115
.83	.045
.80	.075
.81	.065
.81	.065
188.875	0.500

Here the arithmetical mean, or most probable value of the line, is found to be 188.875 meters. The difference between this and the single observations gives the residuals v, whose sum $\Sigma v = 0.5$. Then, by the use of Table IV, for $n = 8$,

$$r = 0.1130 \times 0.5 = 0.0565,$$
$$r_0 = 0.0399 \times 0.5 = 0.0200.$$

By the more accurate formulas (20) and (21) these values are

$$r = 0.051 \quad \text{and} \quad r_0 = 0.018 \text{ meters.}$$

With a larger number of observations, a closer agreement between the probable errors found by the two methods might be expected.

85. The probable error r of a single observation should always be computed, since it furnishes the means of comparing

the accuracy of work done with different instruments, or by different observers. Under similar conditions, r should be practically a constant for a given class of measurements; while for different classes the different values of r indicate the relative precision of the methods. For instance, suppose the same observer to measure the same angle with two different transits, and to find the probable error of a single observation with the first to be 4″, and with the second 6″. The relative precision of the instruments is, then, inversely as these probable errors, or as 3 to 2; and the weights of a single observation in the two cases are as 3^2 to 2^2, or as $2\frac{1}{4}$ to 1; so that one measurement made with the first instrument is worth $2\frac{1}{4}$ made with the second. These results, in order to be satisfactory, must be deduced from a large number of observations; since the formulas for probable error suppose that enough observations are made to exhibit the several residuals according to the law of probability of error as given by equations (1) and (2).

Observations of Unequal Weight.

86. When the observations on a single quantity have different weights, the most probable value of the quantity is to be found by the use of the general arithmetical mean; namely, by multiplying each observation by its weight, and dividing the sum of the products by the sum of the weights. Or if z be that most probable value, M any observation, and p its weight, then, as shown in Art. 45, formula (9) gives

$$z = \frac{\Sigma pM}{\Sigma p}.$$

The probable error of an observation of the weight unity, as shown by formula (24), Art. 67, is

$$r = 0.6745\sqrt{\frac{\Sigma pv^2}{(n-1)}},$$

in which n denotes the number of observations, and v any residual obtained by subtracting M from z. Lastly the probable error of z, as shown in Art. 66, is found by (22),

$$r_0 = \frac{r}{\sqrt{\Sigma p}}.$$

Formula (9) indicates the method of adjustment. Having found the most probable value z, each observation is subtracted from it, giving n residuals v. These are squared, and each v^2 multiplied by the corresponding weight p. The sum of these products is Σpv^2. Then formula (24) gives the probable error of an observation of the weight unity. Lastly, formula (22) gives the probable error of z. And in general the probable error of an observation of given weight may be found by dividing r by the square root of that weight.

87. As an example let the observations in the second column of the following table be the results of the repetition of an angle at different times, 18″.26 arising from five repetitions, 16″.30 from four, and so on, the weights of the observations being taken the same as the number of repetitions. Then the general mean z has the weight 21, the sum of the several weights or

p.	M.	v.	v^2.	pv^2.
5	87° 51′ 18″.26	− 0.10	0.010	0.05
4	16.30	+ 1.86	3.460	13.84
1	21.06	− 2.90	8.410	8.41
4	17.95	+ 0.21	0.044	0.18
3	16.20	+ 1.96	3.842	11.53
4	20.85	− 2.69	7.236	28.94
$\Sigma p = 21$	$z = 87°\ 51'\ 18''.16$			$\Sigma pv^2 = 62.95$

the number of single measures. Subtracting each M from z gives the residuals in the column v; next from Table VI the numbers in the column v^2 are found, and multiplying each of these by the corresponding weight produces the quantities pv^2, whose sum is 62.95. Then, since n is 6, formula (23) gives

$$r = 0.6745\sqrt{\frac{62.95}{5}} = 2''.39,$$

or, by the help of Table III,

$$r = 0.3016\sqrt{62.95} = 2''.39.$$

This is the probable error of an observation of the weight unity. From (22) the probable error of the general mean is,

$$r_0 = \frac{2.39}{\sqrt{21}} = 0''.98,$$

and the probable error of any given observation is found by dividing $2''.39$ by the square root of its weight.

88. The important relation (18) of Art. 63, that the weights of observations are inversely as the squares of their probable errors, furnishes, as already indicated in Art. 85, a ready means of determining weights, if the probable errors can be obtained with sufficient precision. When the weights are known, the observations can be combined by (9), and the most probable value determined.

As an example, consider the two following series of measurements of an angle; the first taken with a transit reading to twenty seconds, and the second with a transit reading to minutes. The angle was observed in each case ten times; the

circle being used in eleven different positions to eliminate errors of graduation, while each time the two verniers were read to eliminate errors of eccentricity.

With First Transit.			With Second Transit.		
M.	*v.*	v^2.	*M.*	*v.*	v^2.
34° 55′ 35″	2	4	34° 56′ 15″	39	1521
35	2	4	55 30	6	36
20	13	169	54 30	66	4356
05	28	784	55 15	21	441
75	42	1764	56 00	24	576
40	7	49	55 45	9	81
10	13	169	55 30	6	36
30	3	9	55 30	6	36
50	17	289	56 00	24	576
30	3	9	55 45	9	81
34° 55′ 33″		3250	34° 55′ 36″		7740

By the method of Art. 80 it is easy to find

For first transit . . . . $34^\circ\ 55'\ 33'' \pm 4''.1$
For second transit . . . . $34\ \ 55\ \ 36\ \ \pm 6\ .3$

Hence by (18) the weights of these means are in the ratio

$$\frac{1}{41^2} : \frac{1}{63^2}, \quad \text{or as 12 to 5 nearly.}$$

The final adjusted value of the angle is, then,

$$z = 34^\circ\ 55' + \frac{33 \times 12 + 36 \times 5}{17} = 34^\circ\ 55'\ 33''.9,$$

and by (18) the probable error of that value is

$$r_0 = 4.1\sqrt{\frac{12}{17}} = 3''.4.$$

As the probable errors of a single observation in the two cases are 13″ and 20″, the corresponding weights are as 400 to 169; so that one observation with the first instrument is worth about $2\frac{1}{3}$ with the second.

When observations upon the same quantity are known to be of different precision, and there is no way of finding the probable errors, as in the example just discussed, weights should be assigned corresponding to the confidence that is placed in them, and then the general mean can be deduced. Of course, the assignment of weights in such cases is a matter requiring experience and judgment.

Problems.

89. The solution of the following problems will serve to exemplify the preceding principles.

1. The latitude of station Bully Spring, on the United States northern boundary, was found by sixty-four observations to be $49° 01' 09''.11 \pm 0''.051$. What was the probable error of a single observation?

2. A line is measured five times, and the probable error of the mean is 0.016 feet. How many additional measurements of the same precision are necessary in order that the probable error of the mean shall be only 0.004 feet?

3. An angle is measured by a theodolite and by a transit with the following results:

By theodolite	$24° 13' 36'' \pm 3''.1$
By transit	$24 \quad 13 \quad 24 \quad \pm 13 .8$

Find the most probable value of the angle and its probable error.

4. A base-line is measured five times with a steel tape reading to hundredths of a foot, and also five times with a chain reading to tenths of a foot, with the following results: —

By the tape:	741.17 feet.	By the chain:	741.2 feet.
	741.09 feet.		741.4 feet.
	741.22 feet.		741.0 feet.
	741.12 feet.		741.3 feet.
	741.10 feet.		741.1 feet.

Find the probable errors and weights for a single observation in the two cases, and also the adjusted length of the line.

Ans. 741.146 ± 0.012.

5. Eight observations of a quantity give the results 769, 768, 767, 766, 765, 764, 763, and 762, whose relative weights are 1, 3, 5, 7, 8, 6, 4, and 2. What is the probable error of the general mean, and the probable error of each observation?

6. The length of a line is stated by one party as 683.4 ± 0.3, and by a second party as 684.9 ± 0.3. What is to be inferred from the two results?

CHAPTER VI.

FUNCTIONS OF OBSERVED QUANTITIES.

90. In this chapter will be discussed the determination of the precision of quantities which are computed from other measured quantities. For instance, the area of a field is a function of its sides and angles: when the most probable values of these have been found by measurement, the most probable value of the area is computed by the rules of geometry, and the precision of that area will depend upon the precision of the measured quantities. Linear measurements will first receive attention; for, although they are direct observations when the result alone is considered, yet really the length of a line is a function of its several parts, namely the sum. So, too, an observed value of an angle is a function (the difference) of two readings. All the following reasoning is based upon the laws of propagation of error deduced in Arts. 68–71.

Linear Measurements.

91. As a line is measured by the continued application of a unit of measure, its probable error should increase with its length. The law of this increase is given by formula (26). If the parts are all equal, and each be taken as the unit of length, the number of parts is the same as the length of the line. Let r denote the probable error of a measurement a unit in length, R the probable error of the total observed length, and l that observed length. Then (26) reduces to

$$R = r\sqrt{l}; \tag{37}$$

that is, the probable error of a measurement of a line increases with the square root of its length.

For example, the value of r for measurements with an engineer's tape on smooth ground is about 0.005: hence, for a line 100 feet long, R is 0.05 feet, and for a line 1,000 feet long, R is 0.16 feet.

Since, by (18), weights are inversely as the squares of probable errors, and, by (37), the squares of probable errors are directly as the lengths of lines, it follows that the weights of linear measurements are inversely as their lengths, or

$$(38) \qquad p_1 : p_2 : p :: \frac{1}{l_1} : \frac{1}{l_2} : \frac{1}{l}.$$

Hence, if the weight of a measurement of a unit's length be 1, the weight of a measurement of the length l will be $\frac{1}{l}$. This principle is to be used in combining linear measurements for which the value of r is the same.

92. The value of r may be found by measuring a line of the length l many times, and computing R by the methods of the last chapter. Then, by (37), the value of r is known. For instance, take the eight measurements of a line about 189 meters long, which are discussed in Art. 84, for which the probable error of a single observation was found to be about 0.05 meters. Here $R = 0.05$, and then $r = \frac{0.05}{\sqrt{189}} = 0.004$ meters, which is the probable error of a measurement of a line one meter in length.

The most convenient way, however, of finding r, is to make duplicate measurements of several lines of different lengths. Let the lengths of the lines be $l_1, l_2 \ldots l_n$, the differences of the duplicate measurements be $d_1, d_2 \ldots d_n$, and the num-

ber of lines be n. These differences are the true errors of a quantity whose true value is zero, and by Art. 67 the probable error of an observed difference is

$$r' = 0.6745\sqrt{\frac{\Sigma pd^2}{n}}.$$

Now, from Art. 68, this probable error is also

$$r' = \sqrt{r^2 + r^2} = r\sqrt{2},$$

and, by equating these two values of r', it is easy to find

$$r = 0.4769\sqrt{\frac{\Sigma pd^2}{n}}, \tag{39}$$

which is the probable error of a measurement a unit long. The weight p is to be taken as $\frac{1}{l}$ in accordance with (38).

For example, the following duplicate measurements of the sides of a mountain field, made with a Gunter's chain, may be considered.

No. of Side.	By First Party.	By Second Party.
1	17.21 chains.	17.18 chains.
2	3.48 "	3.52 "
3	15.14 "	15.19 "
4	1.27 "	1.25 "
5	20.06 "	20.12 "
6	8.85 "	8.92 "
7	0.70 "	0.70 "
8	6.75 "	6.78 "

Here for the first line

$$d_1 = 0.03, \quad d_1^2 = 0.0009, \quad p_1 d_1^2 = \frac{d_1^2}{l_1} = 0.0000523,$$

and similarly for each of the other lines. Then, by addition, $\Sigma pd^2 = 0.001855$, and lastly, from (39), the probable error of a measurement of a unit's length (that is, of one chain) is 0.0073 chains, or 0.73 links.

93. The general formula (26) shows clearly how the precision of linear measurements depends upon the precision of the parts. Evidently the fewer the parts, the smaller will be R, and the greater the precision. Also the longer the chain, the fewer will be the parts, and the greater the precision.

It must be carefully noted, however, that the preceding reasoning only applies to the accidental errors (Art. 7) of observation, and that all constant errors must be investigated, and removed from the results, before the formulas (37) and (39) are used. The effects of temperature on the length of the chain or tape, for instance, may be removed by reading the thermometer, and applying the proper computed corrections, and the effects of side deviations may be removed by making the chain sufficiently longer at the start. In general, the constant errors of linear measurements increase directly as the length of the line; while only the accidental errors increase as the square root of the length.

Angle Measurements.

94. The measurement of an angle is in general effected by taking the difference of readings from a graduated limb; and these readings, in their turn, may be the means of readings on two or more verniers. By the use of the principle expressed in formula (25) it is possible to determine the precision of these readings from the probable errors of observed results.

As an example, the following measurements of an angle made with a transit having two verniers reading to minutes will be discussed. The angle was chosen at about 35° in order that eleven readings might approximately go around the circle, and each reading is the mean of the two verniers.

On Vernier A.	On Vernier B.	Mean Reading.	Angle.
5° 03′ 30″	5° 03′ 30″	5° 03′ 30″	
39 59 30	39 60 00	39 59 45	34° 56′ 15″
74 55 00	74 55 30	74 55 15	55 30
109 49 30	109 50 00	109 49 45	54 30
144 45 00	144 45 00	144 45 00	55 15
179 41 00	179 41 00	179 41 00	56 00
214 37 00	214 36 30	214 36 45	55 45
249 32 30	249 32 00	249 32 15	55 30
284 28 00	284 27 30	284 27 45	55 30
319 24 00	319 23 30	319 23 45	56 00
354 19 30	354 19 30	354 19 30	55 45

By the method of the last chapter it is easy to find that the probable error of a single observation of an angle is nearly 20″. Let r_1 represent the probable error of a reading on one vernier, and r_2 that of the mean of the two verniers. Then by (25), since each observation is the difference of two readings,

$$20 = \sqrt{r_2^2 + r_2^2}, \quad \text{or } r_2 = 14''.1.$$

Next for r_1 the formula (19) gives

$$14.1 = \frac{r_1}{\sqrt{2}}, \quad \text{or } r_1 = 20''.$$

So it appears that the probable error of a single observation of an angle taken in the above manner is the same as that of a

single reading on one vernier. The reading of both verniers not only eliminates the error of eccentricity, but adds much to the precision of the results.

95. By the method of repetitions the precision of angle measures can be further increased. The observations should be conducted like those above described, except that the plate is turned n times between the two readings. Let r_2 be the probable error of a mean reading, and r_3 that of the observed result, which is $\frac{1}{n}$th of the difference of the two readings. Then by (25) and (27), neglecting the error in pointing,

$$r_3 = \frac{r_2}{n}\sqrt{2}.$$

By the method of the last article the mean of n readings would give

$$r_3 = \frac{r_2}{\sqrt{n}}\sqrt{2}.$$

The precision of n repetitions is, hence, $\sqrt{n}$ times greater than the mean of n independent observations. However, the errors in pointing, and other causes, render it doubtful if it is ever advantageous to make n exceed six or eight.

Precision of Areas.

96. Let z_1 and z_2 be the measured sides of a rectangle, and r_1 and r_2 their probable errors. Then by (29) the probable error of the computed area z_1z_2 is

$$R = \sqrt{z_1^2r_2^2 + z_2^2r_1^2}.$$

If r be the probable error of a measurement a unit in length, the law of (37) gives

$$r_1^2 = r^2z_1 \quad \text{and} \quad r_2^2 = r^2z_2;$$

and hence the probable error in the area is

$$R = r\sqrt{z_1 z_2 (z_1 + z_2)}.$$

For instance, let a lot 60×150 feet be laid out by an engineer's chain, for which $r = 0.01$. Then, by the formula, $R = 13.75$ square feet, which is the probable error of 9,000 square feet, the computed area.

97. By the application of formula (30) the probable error of any computed area can be found from the known probable errors of its sides and angles. As one of the simplest cases, take a triangle ABC, whose area is found from the angle A and the two adjacent sides AB and AC. The observed values are

$$\begin{aligned} AB &= 252.52 \pm 0.06, \\ AC &= 300.01 \pm 0.06, \\ A &= 42^\circ 13' 00'' \pm 30''. \end{aligned}$$

The area of this triangle is $\frac{1}{2}AB.AC.\sin A = 25{,}453$ square feet. To compare with (30) let $AB = z_1$, $AC = z_2$, and $\sin A = z_3$; also $r_1 = r_2 = 0.06$, and $r_3 = 0.00011$ = tabular difference corresponding to $30''$. Then

$$\frac{dZ}{dz_1} = \tfrac{1}{2}AC.\sin A,$$
$$\frac{dZ}{dz_2} = \tfrac{1}{2}AB.\sin A,$$
$$\frac{dZ}{dz_3} = \tfrac{1}{2}AB.AC.$$

By inserting all values in (30) it is easy to find $R = 8.9$ square feet for the probable error of the area.

Remarks and Problems.

98. By the application of formulas (25) to (30) the precision of many other functions of observed quantities than those

above noticed may be investigated. A few of the simplest cases are included among the following problems.

1. The radius of a circle is observed as 1000 ± 0.2. Find the probable errors of its circumference and area.

2. Find the maximum probable error of $\sin A + \cos A$ when the probable error of A is $20''$.

3. In order to determine the difference of level between two points A and B, an instrument was set up halfway between them, and twenty readings taken on rods held at each point, with the following results:

Rod at A.	Rod at B.
7 readings gave 7.229 feet.	3 readings gave 9.806 feet.
8 readings gave 7.230 feet.	12 readings gave 9.807 feet.
5 readings gave 7.231 feet.	5 readings gave 9,808 feet.

What is the most probable difference of level between the two points and the probable error of the determination?

Ans. 2.5772 ± 0.00015.

4. A block of cast-iron weighing 100 pounds rests upon a horizontal table, also of cast-iron. A horizontal force is applied to the block, and it is observed that it begins to move when the force is 15.5 pounds. If the probable error in the determination of this force is 0.5 pound, what is the probable error of the co-efficient of friction?

5. A chronometer is rated at a certain date, and found to be $9^m\, 12^s.3$ fast, with a probable error of $0^s.3$. Ten days afterwards it is again rated, and found to be $9^m\, 21^s.4$ fast, with the same probable error. What is the probable error of the mean daily rate?

6. A line of levels is run in the following manner: the back and fore sights are taken at distances of about 200 feet, so that there are thirteen stations per mile, and at each sight the rod is read three times. If the probable error of a single reading is 0.001 feet, what is the probable error of the difference of level of two points which are ten miles apart?

CHAPTER VII.

INDEPENDENT OBSERVATIONS ON SEVERAL QUANTITIES.

99. Independent observations on several related quantities are to be adjusted by the methods of Arts. 46–50, and their precision determined by the methods of Arts. 72–76. The following are the steps of the process:

1st, Let z_1, z_2, z_3, etc., represent the quantities to be determined, and for each observation write an observation equation; or, if more convenient, let z_1, z_2, z_3, etc., be corrections to assumed approximate values of the unknown quantities.

2d, From the observation equations form the normal equations, which will be as many as there are unknown quantities.

3d, Solve the normal equations: the resulting values of the unknown quantities will be their most probable values, that is, the best values that can be deduced from the given observations.

4th, Find the residuals, and the probable error of an observation of the weight unity from formula (32).

5th, Find, if desired, the weights and probable errors of the adjusted values of the unknown quantities.

When the number of unknown quantities exceeds four or five, it will usually be found most convenient to use the algorithm of formulas (10) and (11) for observations of equal weight, and of (12) and (13) for those of unequal weight, and to solve the normal equations by the method of Arts. 51–55. It will, however, probably be best for a beginner to form the normal

equations by the rules in Art. 48 and Art. 50, and to solve them by his own algebraic method.

It will often be convenient to take the unknown quantities as corrections, rather than as the real quantities themselves; since thus the numbers entering the computation are much smaller. The following practical examples will illustrate the whole method of procedure.

Discussion of Level Lines.

100. The following observations are recorded in the Report of the United States Geological and Geographical Survey for 1873, and are here supposed to be of equal reliability or weight:

1. Z_1 above O, 573.08 feet, by Coast Survey and canal levels, via New York and Albany.
2. Z_2 above Z_1, 2.60 feet, by observations on surface of Lake Erie.
3. Z_2 above O, 575.27 feet, by Coast Survey and railroad levels, via New York and Albany.
4. Z_3 above Z_2, 167.33 feet, by railroad levels.
5. Z_4 above Z_3, 3.80 feet, by railroad levels.
6. Z_4 above Z_2, 170.28 feet, by railroad levels, via Alliance.
7. Z_4 above Z_5, 425.00 feet, by railroad levels.
8. Z_5 above O, 319.91 feet, by railroad and Coast Survey levels, via Philadelphia.
9. Z_5 above O, 319.75 feet, by railroad levels, via Baltimore.

The letters here have the following meanings:

O is the mean surface of the Atlantic Ocean.
Z_1 is the mean surface of Lake Erie at Buffalo.
Z_2 is Cleveland city datum plane.
Z_3 is Depot track at Columbus, O.
Z_4 is Union Depot track at Pittsburg.
Z_5 is Depot track at Harrisburg.

It is required to adjust these observations, and to find the probable error of a single observation.

1st, Represent the unknown heights of Z_1, Z_2, Z_3, Z_4, and Z_5 by z_1, z_2, z_3, z_4, and z_5. Then the observations give the observation equations

$$
\begin{aligned}
z_1 &= 573.08,\\
z_2 - z_1 &= 2.60,\\
z_2 &= 575.27,\\
z_3 - z_2 &= 167.33,\\
z_4 - z_3 &= 3.80,\\
z_4 - z_2 &= 170.28,\\
z_4 - z_5 &= 425.00,\\
z_5 &= 319.91,\\
z_5 &= 319.75.
\end{aligned}
$$

2d, Form a normal equation for z_1 by multiplying each equation in which z_1 occurs by its co-efficient in that equation, and adding the products; and in the same way form a normal equation for each of the other unknown quantities. This gives

$$
\begin{aligned}
2z_1 - z_2 \qquad\qquad\qquad &= 570.48,\\
-z_1 + 4z_2 - z_3 - z_4 \qquad &= 240.26,\\
- z_2 + 2z_3 - z_4 \qquad &= 163.53,\\
- z_2 - z_3 + 3z_4 - z_5 &= 599.08,\\
- z_4 + 3z_5 &= 214.66.
\end{aligned}
$$

3d, The solution of these normal equations furnishes the following values:—

$$
\begin{aligned}
z_1 &= 572.81, & z_2 &= 575.14, & z_3 &= 742.05,\\
z_4 &= 745.43, & z_5 &= 320.03,
\end{aligned}
$$

which are the adjusted elevations of the five points above the datum O.

4th, Substitute these values in the observation equations, and find the residuals and their squares; thus:

No.	v.	v^2.
1	0.27	0.073
2	.27	.073
3	.13	.017
4	.42	.176
5	.42	.176
6	.01	.000
7	.40	.160
8	.12	.014
9	.28	.078
		$\Sigma v^2 = 0.767$

Here the number n of observations is 9, and the number q of unknown quantities is 5. The weights p are all unity. Then, from (32),

$$r = 0.6745\sqrt{\frac{0.767}{4}} = 0.295 \text{ feet},$$

which is the probable error of each observation.

5th, To determine the probable errors of the above adjusted values, it is necessary to find their weights by the method of Art. 75. For instance, to find the weight of z_2, represent the absolute term in the normal equation for z_2 by B, and put all the other absolute terms equal to zero. Then the solution gives $z_2 = \frac{26}{51}B$, and accordingly the weight of z_2 is $\frac{51}{26}$. Hence the probable error of the value of z_2 is

$$r_2 = \frac{0.295}{\sqrt{1.96}} = 0.211 \text{ feet};$$

so that the final elevation of Z_2 may be written

$$z_2 = 575.14 \pm 0.21,$$

and it is an even wager that the actual error in the value 575.14 is less (or greater) than the amount 0.21 feet.

101. For level lines of unequal precision the process of adjustment is the same, except, that, before forming the normal equations, each observation equation should be multiplied by the square root of its weight. To illustrate, regard the above nine observations as of unequal weight. The least trustworthy is No. 9; because it is not known that mean tide at Baltimore is the same as the mean surface of the ocean, and its weight may be taken as 1. Nos. 3 to 8 inclusive are ordinary railroad levels, and may, with reference to No. 9, be given a weight of 4. Nos. 1 and 2, being the result of carefully conducted government and canal levels extending over many years, are the most reliable of all; and a weight of 25 may be assigned them. The observation equations are the same as before; multiplying each by the square root of its weight gives

$$\begin{aligned}
5z_1 &= 2865.40,\\
5z_2 - 5z_1 &= 13.00,\\
2z_2 &= 1150.54,\\
2z_3 - 2z_2 &= 334.66,\\
2z_4 - 2z_3 &= 7.60,\\
2z_4 - 2z_2 &= 340.56,\\
2z_4 - 2z_5 &= 850.00,\\
2z_5 &= 639.82,\\
z_5 &= 319.75.
\end{aligned}$$

The normal equations now are

$$\begin{array}{rcl}
50z_1 - 25z_2 & = & 14262.00,\\
-25z_1 + 37z_2 - 4z_3 - 4z_4 & = & 1015.64,\\
-4z_2 + 8z_3 - 4z_4 & = & 654.12,\\
-4z_2 - 4z_3 + 12z_4 - 4z_5 & = & 2396.32,\\
-4z_4 + 9z_5 & = & -100.61,
\end{array}$$

and their solution gives

$$z_1 = 572.98, \qquad z_2 = 575.48, \qquad z_3 = 742.36,$$
$$z_4 = 745.72, \qquad z_5 = 320.25.$$

Inserting these in the observation equations, the remainders or residuals v_1, v_2, etc., are found, and placed in the third column below, their squares in the fourth, and the product of each square by its corresponding weight in the fifth.

No.	p.	v.	v^2.	pv^2.
1	25	0.10	0.010	0.250
2	25	.11	.012	.300
3	4	.20	.040	.160
4	4	.44	.194	.776
5	4	.43	.185	.720
6	4	.02	.000	.002
7	4	.48	.210	.840
8	4	.34	.116	.464
9	1	.50	.250	.250
			$\Sigma pv^2 = 3.762$	

Then by (32) the probable error of an observation of weight unity, that is of No. 9, is

$$r = 0.6745\sqrt{\frac{3.762}{4}} = 0.635 \text{ feet},$$

and the probable error of observations 1 and 2 is by (31)

$$\frac{0.635}{5} = 0.13 \text{ feet},$$

and of those from 3 to 8 inclusive is $\frac{0.635}{2} = 0.32$ feet.

In order, lastly, to find the probable errors of the above adjusted values, their weights must be determined. For instance, to find the weight of z_4, place the absolute term in the fourth normal equation equal to A, and those in the other normal equations equal to zero. Then the solution gives $z_4 = \frac{135}{894}A$, and accordingly the weight of z_4 is 6.62. Hence the probable error of the value of z_4 is

$$r_{z_4} = \frac{0.635}{\sqrt{6.62}} = 0.25 \text{ feet.}$$

And in a similar way the probable error of the value of z_2 may be found to be 0.15 feet.

102. For such simple cases as those just presented, the absolute terms in the normal equations might be represented by letters, A_1, A_2, etc., and a general solution easily effected, which would give at once all the weights and unknown quantities. For instance, if the normal equations of Art. 100 are thus written

$$\begin{aligned}
2z_1 - z_2 \qquad\qquad\qquad\qquad &= A_1,\\
-z_1 + 4z_2 - z_3 - z_4 \qquad\quad &= A_2,\\
- z_2 + 2z_3 - z_4 \qquad\quad &= A_3,\\
- z_2 - z_3 + 3z_4 - z_5 &= A_4,\\
- z_4 + 3z_5 &= A_5,
\end{aligned}$$

the solution gives

$$\begin{aligned}
51z_1 &= 32A_1 + 13A_2 + 11A_3 + 9A_4 + 3A_5,\\
51z_2 &= 13A_1 + 26A_2 + 22A_3 + 18A_4 + 6A_5,\\
51z_3 &= 11A_1 + 22A_2 + 50A_3 + 27A_4 + 9A_5,\\
17z_4 &= 3A_1 + 6A_2 + 9A_3 + 12A_4 + 4A_5,\\
17z_5 &= A_1 + 2A_2 + 3A_3 + 4A_4 + 7A_5,
\end{aligned}$$

where all the weights are at once seen, and from which the values of the unknown quantities can easily be found.

As indicated in Art. 99, the numerical operations may be somewhat simplified by taking the unknown quantities as corrections to be applied to assumed elevations of Z_1, Z_2, etc. Thus it is seen from the observations that 573 and 575 feet are approximate elevations for Z_1 and Z_2. By writing, then,

$$\begin{aligned}
\text{elevation of } Z_1 &= 573 + z_1,\\
\text{elevation of } Z_2 &= 575 + z_2,\\
\text{elevation of } Z_3 &= 742 + z_3,\\
\text{elevation of } Z_4 &= 745 + z_4,\\
\text{elevation of } Z_5 &= 320 + z_5,
\end{aligned}$$

the following simpler observation equations are obtained from the given data:

$$\begin{aligned}
z_1 &= 0.08,\\
z_2 - z_1 &= 0.60,\\
z_2 &= 0.27,\\
z_3 - z_2 &= 0.33,\\
z_4 - z_3 &= 0.80,\\
z_4 - z_2 &= 0.28,\\
z_4 - z_5 &= 0.00,\\
z_5 &= -0.09,\\
z_5 &= -0.25.
\end{aligned}$$

From these the normal equations are formed, whose first members are the same as written above, and whose second members have the values $A_1 = -0.52$, $A_2 = +0.26$, $A_3 = -0.47$, $A_4 = +1.08$, $A_5 = -0.34$. The solution of the normal equations gives

$$z_1 = -0.19, \quad z_2 = 0.14, \quad z_3 = 0.05, \quad z_4 = 0.43, \quad z_5 = 0.03;$$

and the final elevations are

$$\begin{aligned}
Z_1 &= 573.00 - 0.19 = 572.81,\\
Z_2 &= 575.00 + 0.14 = 575.14, \quad \text{etc.},
\end{aligned}$$

which are the same as found in Art. 100.

Angles at a Station.

103. When two angles and also their sum are observed at a station, the observed sum usually differs from the sum of the two measured single angles. Let the observation of the first angle give the result M_1, of the second M_2, and that of their sum M_3. Then $M_1 + M_2$ is greater or less than M_3 by a certain discrepancy d. It is required to adjust the observations, regarding the weights as equal, and to find the probable errors of the adjusted values.

1st, Let z_1 and z_2 be the most probable corrections to the observed values M_1 and M_2, so that $M_1 + z_1$ and $M_2 + z_2$ are the most probable values of the first and second angles. The observation equations then are

$$\begin{aligned} M_1 + z_1 &= M_1, \\ M_2 + z_2 &= M_2, \\ (M_1 + z_1) + (M_2 + z_2) &= M_3, \end{aligned}$$

which reduce to

$$\begin{aligned} z_1 &= 0, \\ z_2 &= 0, \\ z_1 + z_2 &= M_3 - (M_1 + M_2) = d. \end{aligned}$$

2d, From these, the normal equations are

$$\begin{aligned} 2z_1 + z_2 &= d, \\ z_1 + 2z_2 &= d. \end{aligned}$$

3d, The solution of the normal equations gives

$$z_1 = \tfrac{1}{3}d, \quad \text{and } z_2 = \tfrac{1}{3}d,$$

for the most probable values of the corrections: hence the adjusted values are

$$\begin{aligned} &M_1 + \tfrac{1}{3}d, \\ &M_2 + \tfrac{1}{3}d, \\ &M_3 - \tfrac{1}{3}d. \end{aligned}$$

4th, The residuals are evidently the three corrections, the sum of whose squares is $\frac{1}{3}d^2$; then, from (32),

$$r = 0.6745\sqrt{\tfrac{1}{3}d^2} = 0.389d,$$

which is the probable error of a single observed value.

5th, By the method of Art. 75 it is easy to find that the weights of the adjusted values of z_1 and z_2 are 1.5: hence their probable errors are

$$\frac{0.389d}{\sqrt{1.5}} = 0.318d,$$

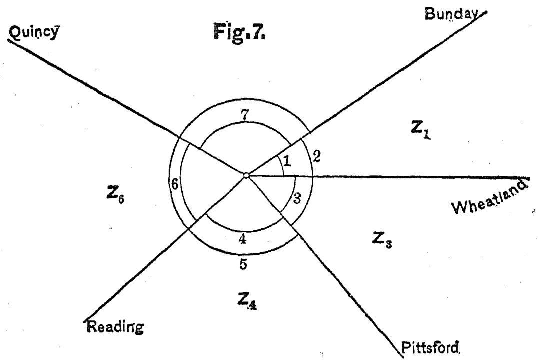
and evidently the probable error of the adjusted value of $z_1 + z_2$ is also $0.318d$.

104. When several angles are observed at a station, several sums and differences of simple angles are often taken. For example, the following are the angles observed at the Station Hillsdale, on the United States Lake Survey; each being the mean of nearly the same number of readings, and hence regarded as of the same weight. (See Report of United States Lake Survey, p. 449.)

No.	Between Stations.	Observation.
1	Bunday and Wheatland	44° 25′ 40″.613
2	Bunday and Pittsford	80 47 32.819
3	Wheatland and Pittsford	36 21 51.996
4	Pittsford and Reading	91 34 24.758
5	Pittsford and Bunday	279 12 27.619
6	Reading and Quincy	62 37 43.405
7	Quincy and Bunday	125 00 18.808

The annexed figure shows the relative positions of the stations and of the seven observed angles. It is required to adjust the observed results, and to find their probable errors.

Fig. 7.



1st, Let Z_1, Z_3, Z_4, and Z_6 be the required most probable values of four of the simple angles as indicated in Fig. 7; then the observation equations are

$$\begin{aligned}
Z_1 &= 44^\circ\ 25'\ 40''.613,\\
Z_1 + Z_3 &= 80\ \ 47\ \ 32.819,\\
Z_3 &= 36\ \ 21\ \ 51.996,\\
Z_4 &= 91\ \ 34\ \ 24.758,\\
360^\circ - (Z_1 + Z_3) &= 279\ \ 12\ \ 27.619,\\
Z_6 &= 62\ \ 37\ \ 43.405,\\
360^\circ - (Z_1 + Z_3 + Z_4 + Z_6) &= 125\ \ 00\ \ 18.808.
\end{aligned}$$

Assume the measured values of Z_1, Z_3, Z_4, and Z_6 as approximate, and let z_1, z_3, z_4, and z_6 be the most probable corrections, thus

$$\begin{aligned}
Z_1 &= 44^\circ\ 25'\ 40''.613 + z_1,\\
Z_3 &= 36\ \ 21\ \ 51.996 + z_3,\\
Z_4 &= 91\ \ 34\ \ 24.758 + z_4,\\
Z_6 &= 62\ \ 37\ \ 43.405 + z_6.
\end{aligned}$$

Then, by inserting these values in the observation equations, the following simpler observation equations are found :

$$
\begin{aligned}
z_1 &= \quad 0, \\
z_1 + z_3 &= +0.210, \\
z_3 &= \quad 0, \\
z_4 &= \quad 0, \\
z_1 + z_3 &= -0.228, \\
z_6 &= \quad 0, \\
z_1 + z_3 + z_4 + z_6 &= +0.420,
\end{aligned}
$$

in which the right-hand members denote seconds only.

2d, The normal equations are now easily written, either by the rule of Art. 48, or by the help of the algorithm of formulas (10) and (11). They are

$$
\begin{aligned}
4z_1 + 3z_3 + z_4 + z_6 &= +0.402, \\
3z_1 + 4z_3 + z_4 + z_6 &= +0.402, \\
z_1 + z_3 + 2z_4 + z_6 &= +0.420, \\
z_1 + z_3 + z_4 + 2z_6 &= +0.420.
\end{aligned}
$$

3d, The solution of these equations gives

$$z_1 = z_3 = +0''.022, \qquad z_4 = z_6 = 0''.126.$$

The addition of these corrections to the approximate values gives the most probable values of the angles Nos. 1, 3, 4, and 6; and from these, by simple addition, the most probable values of Nos. 2, 5, and 7, are obtained. Thus, the adjusted values are

$$
\begin{aligned}
\text{No. } 1 &= 44^\circ\ 25'\ 40''.635 = Z_1, \\
\text{No. } 3 &= 36\ \ \ 21\ \ \ 52.018 = Z_3, \\
\text{No. } 4 &= 91\ \ \ 34\ \ \ 24.884 = Z_4, \\
\text{No. } 6 &= 62\ \ \ 37\ \ \ 43.531 = Z_6, \\
\text{No. } 2 &= 80\ \ \ 47\ \ \ 32.653 = Z_1 + Z_3, \\
\text{No. } 5 &= 279\ \ \ 12\ \ \ 27.347 = 360^\circ - (Z_1 + Z_3), \\
\text{No. } 7 &= 125\ \ \ 00\ \ \ 18.932 = 360^\circ - (Z_1 + Z_3 + Z_4 + Z_6).
\end{aligned}
$$

4th, The differences between the observed and the adjusted values are the residuals, which, with their squares, are thus arranged:

No.	Observed.	Adjusted.	v.	v^2.
1	40″.613	40″.635	+ 0.022	0.0005
2	32.819	32.653	− 0.166	.0276
3	51.996	52.018	+ 0.022	.0005
4	24.758	24.884	+ 0.126	.0159
5	27.619	27.347	− 0.272	.0740
6	43.405	43.531	+ 0.126	.0159
7	18.808	18.932	+ 0.124	.0154

The sum Σv^2 is here 0.1498; and hence, by formula (32), the probable error of a single observation is

$$r = 0.6745\sqrt{\frac{0.1498}{3}} = 0''.151.$$

5th, By writing A for the absolute term in the first normal equation, and zero for the absolute terms in the other normal equations, the solution gives the value of z_1 as $\frac{10}{17}A$; and hence the weight of z_1 is 1.7. In a similar way the weight of z_4 is found to be 1.4. The probable errors of the adjusted values of z_1 and z_3 are now

$$\frac{0.151}{\sqrt{1.7}} = 0''.116;$$

and those of the adjusted values of z_4 and z_6 are

$$\frac{0.151}{\sqrt{1.4}} = 0''.128.$$

In order to find the probable errors of angles Nos. 2, 5, and 7, it would be necessary to represent them by single letters, and to form and solve another set of normal equations.

105. As an example of angles with unequal weights, the following observations at North Base, Keweenaw Point, on the United States Lake Survey, will next be considered:

No.	Between Stations.	Observed Angle.	Weight.
1	Crebassa and Middle	55° 57′ 58″.68	3
2	Middle and Quaquaming	48 49 13.64	19
3	Crebassa and Quaquaming	104 47 12.66	17
4	Quaquaming and South Base	54 38 15.53	13
5	Middle and South Base	103 27 28.99	6

Let Z_1, Z_2, and Z_4 represent the angles Nos. 1, 2, and 4; then the observation equations are

$$\begin{aligned} Z_1 &= 55^\circ\ 57'\ 58''.68, && \text{with weight } 3, \\ Z_2 &= 48\ \ 49\ \ 13.64, && \text{with weight } 19, \\ Z_1 + Z_2 &= 104\ \ 47\ \ 12.66, && \text{with weight } 17, \\ Z_4 &= 54\ \ 38\ \ 15.53, && \text{with weight } 13, \\ Z_2 + Z_4 &= 103\ \ 27\ \ 28.99, && \text{with weight } 6. \end{aligned}$$

Let z_1, z_2, and z_4 be corrections to the measured values of Z_1, Z_2, and Z_4; then the simpler observation equations are

$$\begin{aligned} z_1 &= 0, && \text{with weight } 3, \\ z_2 &= 0, && \text{with weight } 19, \\ z_1 + z_2 &= +0.34, && \text{with weight } 17, \\ z_4 &= 0, && \text{with weight } 13, \\ z_2 + z_4 &= -0.18, && \text{with weight } 6. \end{aligned}$$

From these, the normal equations are formed, either by the rule of Art. 50, or by the help of the algorithm of formulas (12) and (13). They are

$$\begin{aligned} 20z_1 + 17z_2 \qquad\quad &= +5.78, \\ 17z_1 + 42z_2 + \ \ 6z_4 &= +4.70, \\ 6z_2 + 19z_4 &= -1.08. \end{aligned}$$

The solution of these equations gives

$$z_1 = +0''.285, \qquad z_2 = +0''.005, \qquad z_4 = -0.059.$$

Hence the following are the adjusted angles

$$\begin{aligned} \text{No. } 1 &= \ \ 55^\circ \ 57' \ 58''.965, \\ \text{No. } 2 &= \ \ 48 \quad 49 \quad 13.645, \\ \text{No. } 3 &= 104 \quad 47 \quad 12.610, \\ \text{No. } 4 &= \ \ 54 \quad 38 \quad 15.471, \\ \text{No. } 5 &= 103 \quad 27 \quad 29.116. \end{aligned}$$

To find the probable errors, the residuals are next obtained.

No.	Observed.	Adjusted.	v.	v^2.	p.	pv^2.
1	58″.68	58″.965	+ 0.285	0.0812	3	0.244
2	13.64	13.645	+ 0.005	.0000	19	.000
3	12.66	12.610	− 0.050	.0025	17	.042
4	15.53	15.471	− 0.059	.0035	13	.045
5	28.99	29.116	+ 0.126	.0159	6	.095

The sum Σpv^2 is here 0.426; then, by (32),

$$r = 0.6745\sqrt{\frac{0.426}{2}} = 0''.31,$$

which is the probable error of an observation of the weight unity. The probable error of the observed angle, No. 2, is, then,

$$r_2 = \frac{0.31}{\sqrt{19}} = 0''.07.$$

The probable error of the final value of No. 2 must be less than $0''.07$, since its weight is increased by the adjustment.

Empirical Constants.

106. One of the most important applications of the Method of Least Squares is the deduction, from observations, of the values of physical constants or co-efficients. In all such cases a theoretical formula or law is first established, which contains the co-efficients in a literal form; and this law serves to state as many observation equations as there are observations. The method of procedure is then exactly the same as that outlined in the first article of this chapter. The precision of the values deduced for the constants depends, of course, upon the precision and number of the observations which enter the discussion.

As an example, take the determination of the ellipticity of the earth by means of experiments on the length of the seconds' pendulum. In 1743 Clairaut deduced the following remarkable law:

$$s = S + S(\tfrac{5}{2}k - f)\sin^2 l,$$

in which S is the length of the seconds' pendulum at the equator, and s its length at any latitude l, while k is the ratio of the centrifugal force at the equator to gravity, and f is the fraction expressing the ellipticity of the earth. This may be written

$$s = S + T\sin^2 l.$$

Now, by measuring s at two different latitudes, two equations would result, from which values of S and T could be found; and, by measuring s at many different latitudes, many equations would result, from which the most probable values of S and T may be found. The following, for instance, are thirteen observations, taken by Sabine in the years 1822–24:

Place.	Latitude.	Length of Seconds' Pendulum.
		English Inches.
Spitzbergen	$+79° 49' 58''$	39.21469
Greenland	74 32 19	39.20335
Hammerfest	70 40 5	39.19519
Drontheim	63 25 54	39.17456
London	51 31 8	39.13929
New York	40 42 43	39.10168
Jamaica	17 56 7	39.03510
Trinidad	10 38 56	39.01884
Sierra Leone	8 29 28	39.01997
St. Thomas	0 24 41	39.02074
Maranham	−2 31 43	39.01214
Ascension	7 55 48	39.02410
Bahia	12 59 21	39.02425

For each of these an observation equation is now to be written. Thus, for the first,

$$s = 39.21469.$$
$$l = 79° 49' 58''.$$
$$\sin l = 0.9842965.$$
$$\sin^2 l = 0.9688402.$$
$$39.21469 = S + 0.9688402\,T.$$

And in like manner the following thirteen observation equations are stated:

$$
\begin{aligned}
39.21469 &= S + 0.9688402T. \\
39.20335 &= S + 0.9289304T. \\
39.19519 &= S + 0.8904120T. \\
39.17456 &= S + 0.7999544T. \\
39.13929 &= S + 0.6127966T. \\
39.10168 &= S + 0.4254385T. \\
39.03510 &= S + 0.0948286T. \\
39.01884 &= S + 0.0341473T. \\
39.01997 &= S + 0.0218023T. \\
39.02074 &= S + 0.0000515T. \\
39.01214 &= S + 0.0019464T. \\
39.02410 &= S + 0.0190338T. \\
39.02425 &= S + 0.0505201T.
\end{aligned}
$$

The normal equations formed from these are

$$
\begin{aligned}
508.18390 &= 13.000000S + 4.848702T, \\
189.94447 &= 4.848702S + 3.704394T,
\end{aligned}
$$

whose solution gives

$$
\begin{aligned}
S &= 39.01568 \text{ inches}, \\
T &= 0.20213 \text{ inches},
\end{aligned}
$$

as the most probable values that can be deduced from the thirteen observations. Hence the length of the seconds' pendulum at any latitude, l, may be written

$$s = 39.01568 + 0.20213 \sin^2 l.$$

The values thus deduced for S and T are empirical constants. To find from them the ellipticity f, it is easily seen that

$$f = \tfrac{5}{2}k - \frac{T}{S},$$

and, as the value of k is known to be $\frac{1}{289}$, that of f is

$$f = 0.0086505 - 0.0051807 = \frac{1}{288.2}.$$

If desired, the precision of the constants S and T may be investigated by determining their weights and probable errors, and from these the precision of the value of f may also be inferred.

107. When two quantities x and y are connected by the relation $y = Sx + T$ the method of the last article can, in strictness, only be applied to find the most probable values of S and T when the observed values of x are free from error. If x is liable to error as well as y, the following method may be used.* First let the value of S be found, supposing that x is without error, and let this be called S_1. Secondly, let the value of S be found regarding y as without error, and let this be called S_2. Let each observed value of x have the weight p, and each observed value of y have the weight unity. Then the most probable value of S is found by solving the quadratic equation

$$S^2 - \left(S_2 - \frac{p}{S_1}\right) S - p = 0,$$

and, if n be the number of pairs of observations, the formula

$$T = \frac{1}{n}\left(\Sigma y - S.\Sigma x\right)$$

gives the most probable value of T. The following numerical example will illustrate the method.

In order to determine the most probable equation of a cer-

* Report U. S. Coast and Geodetic Survey, 1890, p. 687.

tain straight line the abscissas and ordinates of four of its points were measured with equal precision, giving

$$y = 0.5, \quad 0.8, \quad 1.0, \quad \text{and} \quad 1.2.$$
$$x = 0.4, \quad 0.6, \quad 0.8, \quad \text{and} \quad 0.9.$$

First, supposing that the values of x are without error, the four observation equations are written:—

$$0.5 = 0.4S + T,$$
$$0.8 = 0.6S + T,$$
$$1.0 = 0.8S + T,$$
$$1.2 = 0.9S + T.$$

And then, forming and solving the normal equations, there is found $S_1 = 1.339$. Secondly, supposing that the values of y are without error, the equation of the line must be written in the form

$$x = \frac{y}{S} - \frac{T}{S} = Uy + V,$$

and the observation equations are

$$0.4 = 0.5\,U + V,$$
$$0.6 = 0.8\,U + V,$$
$$0.8 = 1.0\,U + V,$$
$$0.9 = 1.2\,U + V;$$

from which the normal equations are derived, and by their solution $U = 0.7385$, or $S_2 = 1.354$.

These values of S_1 and S_2 give the quadratic equation $S^2 - 0.607S - 1 = 0$, whence $S = 1.348$, and then T is found to be -0.035, and accordingly

$$y = 1.348x - 0.035$$

is the most probable equation of the line as derived from the four pairs of observations.

107′. The determination of the elements of the orbit of a comet or planet is another instance of the deduction of empirical constants. Here the observed quantities are the right

ascension and declination of the body at various points in its orbit. Through any three of these points a curve may be passed, and an orbit computed by the formulas of theoretical astronomy. The problem, however, is to determine the most probable orbit by the use of all the observations.

The first step, after collecting and reducing the observations, is to select a few favorably situated, and from them to compute the approximate elements of an elliptical or parabolic orbit, as the case may require. With these approximate elements, the places of the body are computed for as many dates as there are observations, and the differences between the computed and observed places found. A theoretic differential formula is next established for a difference in right ascension, and another for a difference in declination, in terms of unknown corrections to the assumed elements, and of co-efficients that may be computed from the observations. Each observation thus furnishes a difference, and each difference an observation equation, whose unknown quantities are the corrections to the approximate elements of the orbit. From the observation equations the normal equations are derived and solved, and the most probable set of corrections found. Lastly, the application of these corrections to the approximate elements furnishes the most probable elements that can be deduced from the given observations.

The process thus briefly described is very lengthy in its actual application. For instance, in Hall's determination of the elements of the orbit of the outer satellite of Mars * there are forty-nine observation equations, each containing seven unknown corrections, and forty-nine others, each containing six. From these the seven normal equations were formed, and by their solution the most probable values found for the corrections. The precision of the elements of the orbit was also deduced by computing the probable errors of the corrections.

* Hall's Observations and Orbits of the Satellites of Mars; Washington, 1878.

Empirical Formulas.

108. The case of the last article is that of a rational formula with empirical constants. An empirical formula, on the other hand, is one assumed to represent certain observations, and which is not known to express the law governing them. The constants in such formulas are also best determined by the application of the Method of Least Squares.

The first step in the establishment of an empirical formula is to plot the given observations, taking one observed quantity as abscissas, and the other as ordinates. Let y and x be the two quantities between which an empirical formula is to be established. The plot shows to the eye how y varies with x. If y is a continually increasing function of x, or if the curve resembles a parabola, the general equation

$$y = S + Tx + Ux^2 + Vx^3 + \text{etc.}, \tag{40}$$

may be written to represent the relation between y and x. This equation applies to a large class of physical phenomena, such as relations between space and velocity, volume and temperature, stress and strain, and other similar related quantities. The letters S, T, U, etc., represent constants whose values are to be determined from the observations.

Another large class of phenomena may be represented by the general equation

$$\begin{aligned} y = S + T \sin \frac{360^\circ}{m} x + T' \cos \frac{360^\circ}{m} x \\ + U \sin \frac{360^\circ}{m} 2x + U' \cos \frac{360^\circ}{m} 2x + \text{etc.}, \end{aligned} \tag{41}$$

in which, as x increases, y passes through repeating cycles. As such may be mentioned the variation of temperature throughout the year, the changes of the barometer, the ebb and flow of the tides, the distribution of heat on the surface of the earth depending on latitude, and, in fact, all phenomena which repeat

themselves like the oscillations of a pendulum. The letters S, T, U, etc., represent constants which are to be found from the observations; while m is the number of equal parts into which the whole cycle is divided, and must be taken in terms of the same unit as x. If the several cycles are similar and regular, only the first three terms are required to represent the variation.

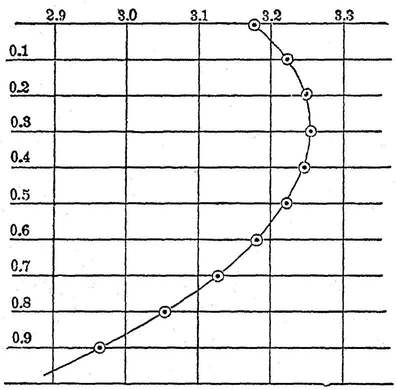
Other general empirical formulas than (40) and (41) are also employed in discussing physical phenomena. Exactly what formula will apply to a given set of observations, so as to agree well with them, and at the same time be of use in other similar cases, can only be determined by trial. The investigator must, from his knowledge of physical laws, assume such an expression as seems most plausible, and then deduce the most probable values of the constants. The comparison of the observed and calculated results furnishes the residuals, from which, if desired, the probable errors may be deduced. When several empirical formulas have been determined for the same observations, that one is the best which furnishes the smallest value for the sum of the squares of the residuals.

109. Consider as a first practical example the deduction of the equation of the vertical velocity curve for the observations given on p. 244 of the second edition of the "Report on the Physics and Hydraulics of the Mississippi River," by Humphreys and Abbot. The grand means of the measurements give the following results for the velocities at different depths below the surface:

At surface,	3.1950 feet per second.
At 0.1 depth,	3.2299 feet per second.
At 0.2 depth,	3.2532 feet per second.
At 0.3 depth,	3.2611 feet per second.
At 0.4 depth,	3.2516 feet per second.
At 0.5 depth,	3.2282 feet per second.
At 0.6 depth,	3.1807 feet per second.
At 0.7 depth,	3.1266 feet per second.
At 0.8 depth,	3.0594 feet per second.
At 0.9 depth,	2.9759 feet per second.

These observations may be plotted by dividing a vertical line representing the depth of the river into ten equal parts, through the points of division drawing horizontal lines, and laying off upon these the observed velocities. On the annexed figure the points enclosed within small circles represent the observations. Each horizontal division of the diagram is 0.1 feet per second, and each vertical division is one-tenth of the depth.

Fig. 8.



Let y be the velocity at any point whose depth below the surface is x, the total depth of the river being unity, and assume that three terms of formula (40) will give the relation between y and x, or that

$$y = S + Tx + Ux^2.$$

This is equivalent to assuming that the curve of vertical velocities is a parabola, with its axis horizontal.

The observations furnish the values of y for ten values of x; and thus, for determining S, T, and U, there are the following ten observation equations:—

$$\begin{aligned}
3.1950 &= S + 0.0T + 0.00U. \\
3.2299 &= S + 0.1T + 0.01U. \\
3.2532 &= S + 0.2T + 0.04U. \\
3.2611 &= S + 0.3T + 0.09U. \\
3.2516 &= S + 0.4T + 0.16U. \\
3.2282 &= S + 0.5T + 0.25U. \\
3.1807 &= S + 0.6T + 0.36U. \\
3.1266 &= S + 0.7T + 0.49U. \\
3.0594 &= S + 0.8T + 0.64U. \\
2.9759 &= S + 0.9T + 0.81U.
\end{aligned}$$

From these the following three normal equations are found:

$$\begin{aligned}31.761600 &= 10.00S + 4.500T + 2.8500U.\\ 14.089570 &= 4.50S + 2.850T + 2.0250U.\\ 8.828813 &= 2.85S + 2.025T + 1.5333U.\end{aligned}$$

And their solution gives

$$S = +3.19513, \qquad T = +0.44253, \qquad U = -0.7653.$$

Accordingly, the empirical formula of vertical velocities is

$$y = 3.19513 + 0.44253x - 0.7653x^2,$$

where y is the velocity in feet per second at any decimal depth x. The curve corresponding to this formula is drawn on the above diagram.

The following is a comparison of the observed velocities with those computed from this empirical formula:

x.	Observed y.	Computed y.	v.	v^2.
0.0	3.1950	3.1951	− 0.0001	0.000000
0.1	3.2299	3.2317	− 0.0018	3
0.2	3.2532	3.2530	+ 0.0002	0
0.3	3.2611	3.2590	+ 0.0021	4
0.4	3.2516	3.2497	+ 0.0019	4
0.5	3.2282	3.2251	+ 0.0031	10
0.6	3.1807	3.1851	− 0.0044	19
0.7	3.1266	3.1299	− 0.0033	11
0.8	3.0594	3.0594	0.0000	0
0.9	2.9759	2.9735	+ 0.0024	6
1.0		2.8724		

The sum of the squares of the residuals is here 0.000057, and hence

$$r = 0.6745\sqrt{\frac{0.000057}{10 - 3}} = 0.0019$$

is the probable value of a residual, or the probable difference between an observed and computed velocity. The agreement between the parabola and the observed points is very close.*

110. As a second example, consider the deduction of a formula to express the magnetic declination at Hartford, Conn., for which place the following observations are given on p. 225 of the United States Coast and Geodetic Survey Report for 1882:

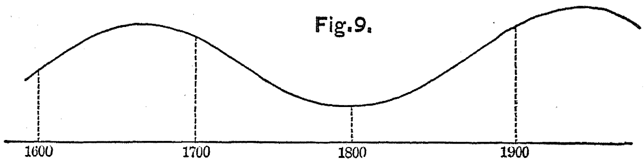
Date.	Declination.	
1786	5°	25′ W.
1810	4	46
1824	5	45
1828–29	6	03
27 July, 1859	7	17
16 Aug., 1867	7	49.3
25 July, 1879	8	34.0

From numerous records at various places, it is known that the declination oscillates slowly to and fro, passing through a cycle in a period varying, at different places, from two hundred and fifty to four hundred years. The variation in New England

* See further, concerning this curve, in Journal Franklin Institute, 1877, vol. civ, p. 233; also Van Nostrand's Magazine, 1877, vol. xvii, p. 443, and 1878, vol. xviii, p. 1. The reasoning of Hagen concerning the probable errors on p. 447 of the second article is thought to be incorrect.

may be roughly represented by the annexed figure, where the ordinates to the curve show the relative values of the declination at the respective years. Formula (41) is hence applicable to the discussion of the above observations.

Fig. 9.



Let y be the magnetic declination at the time x, and assume the empirical relation

$$y = S + T \sin \frac{360^\circ}{m} x + T' \cos \frac{360^\circ}{m} x.$$

Here there are four constants, S, T, T', and m, to be found by the Method of Least Squares from the given observations. The only practical way of procedure is to assume a plausible value of m, and then to state the observation equations and normal equations, from which values of S, T, and T' may be deduced. Again: assume another value of m, and repeat the computation, thus finding other values for S, T, and T'. If necessary, the computation is to be repeated for several values of m; and for each formula thus deduced the residuals, or differences between the observed and computed values of y, are to be found. Then that value of m and that formula is the best which makes the sum of the squares of the residuals a minimum.

Take for m the value 288 years; then $\frac{360^\circ}{m}$ is 1.25, and the formula is

$$S + T \sin 1.25x + T' \cos 1.25x = y.$$

Let x be the number of years counted from the epoch, Jan. 1,

1850, and let all angles be expressed in degrees and decimals; then, for the first observation,

$$\begin{aligned} x &= 1786.5 - 1850.0 = -63.5 \text{ years}, \\ 1.25x &= -79.4 \text{ degrees}, \\ \sin 1.25x &= -0.983, \\ \cos 1.25x &= +0.184, \\ y &= 5.42 \text{ degrees}, \end{aligned}$$

and hence the first observation equation is

$$S - 0.983T + 0.184T' = 5.42.$$

In like manner the following tabulation is made

No.	Date.	x.	$1.25x$.	Sin $1.25x$.	Cos $1.25x$.	y.
1	1786.5	−63.5	−79°.4	−0.983	+0.184	+5°.42
2	1810.5	−39.5	−49.4	−0.759	+0.651	+4.77
3	1824.5	−25.5	−31.9	−0.528	+0.849	+5.75
4	1829.0	−21.0	−26.25	−0.442	+0.897	+6.05
5	1859.6	+ 9.6	+12.0	+0.208	+0.978	+7.29
6	1867.6	+17.6	+22.0	+0.375	+0.927	+7.82
7	1879.6	+29.6	+37.0	+0.602	+0.799	+8.57

From the last three columns the seven observation equations are written; and from these the three normal equations are easily formed, either by the rule of Art. 48, or by the help of the algorithm of formulas (10) and (11). They are

$$\begin{aligned} +7.00S - 1.53T + 5.28T' - 45.67 &= 0, \\ -1.53S + 2.56T - 0.51T' + 5.03 &= 0, \\ +5.28S - 0.51T + 4.53T' - 35.64 &= 0, \end{aligned}$$

and their solution gives

$$S = +8^\circ.06, \quad T = +2^\circ.60, \quad T' = -1^\circ.29.$$

Hence the empirical formula is

$$y = 8^\circ.06 + 2^\circ.60 \sin 1.25x - 1^\circ.29 \cos 1.25x.$$

This may also be written

$$y = +8^\circ.06 + 2^\circ.90 \sin (1^\circ.25x - 26^\circ.4),$$

which is a more convenient form for discussion.*

The following is a comparison of the observed declinations with those computed from this formula:

Date.	x.	Observed y.	Computed y.	v.
1786.5	−63.5	+5°.42	5°.28	+0.14
1810.5	−39.5	4.77	5.25	−0.48
1824.5	−25.5	5.75	5.60	+0.15
1829.0	−21.0	6.05	5.76	+0.29
1859.6	+ 9.6	7.29	7.34	−0.05
1867.6	+17.6	7.82	7.84	−0.02
1879.6	+29.6	8.57	8.59	−0.02

The sum of the squares of these residuals is 0.36, and hence, by (32),

$$r = 0.6745 \sqrt{\frac{0.36}{7-3}} = 0.19,$$

which gives the probable error of a single computed value if the observations be regarded as exact, or the probable error of an observation if the law expressed in the empirical formula be regarded as exact.

* See the numerous valuable papers by Schott, in the Reports of the United States Coast and Geodetic Survey, the latest of which is in the Report for 1882, pp. 211–276. The above formula for the declination is the one there adopted, as giving the best value of the period m.

111. Lastly, consider the deduction of a formula to represent certain experiments, made by Darcy and Bazin, on the flow of water in a rectangular wooden trough lined with cement. The width of the trough was 1.812 meters, and its slope 0.0049. Water was allowed to run through it with varying depths; and for each depth the mean velocity was measured, and the hydraulic radius of the water-section computed by dividing the wetted perimeter into the area of the section. The following are the results, the hydraulic radius h being given in meters, and the mean velocity m in meters per second:

No.	h.	m.
1	0.1144	1.731
2	.1312	1.853
3	.1445	1.984
4	.1579	2.081
5	.1701	2.171
6	.1813	2.258
7	.1925	2.326
8	.2026	2.397
9	.2123	2.460

Assume the expression

$$m = sh^t,$$

and let it be required to find from the above experiments the most probable values of s and t. First reduce the expression to a linear form by writing it thus:

$$\log m = \log s + t \log h.$$

Each observation furnishes an observation equation containing $\log s$ and t. For example, the first is

$$0.2383 = \log s - 0.9416t.$$

The twelve observation equations furnish the two normal equations, and their solution gives

$$t = 0.572, \quad \log s = 0.7767, \quad \therefore s = 5.98.$$

Therefore the empirical formula

$$m = 5.98 h^{0.572}$$

is the best of the assumed form that can be derived from the nine experiments.

112. *Problems.*

1. The following levels were taken to determine the elevations of five points, T, U, W, X, and Y, above the datum O:

T above $O = 115.52$.
U above $T = 60.12$.
U above $O = 177.04$.
W above $T = 234.12$.
W above $U = 171.00$.
X above $W = 632.25$.
X above $Y = 211.01$.
Y above $U = 596.12$.
Y above $W = 427.18$.

What are the adjusted elevations?

Ans. $T = 115.61$, $U = 176.95$, etc.

2. Four angles are observed at a station, and also their sum. The observed sum differs from the sum of the four observed parts by the discrepancy d. What are the adjusted values?

3. Adjust the following angles, taken at the station Moodus, and find the probable errors of the adjusted values.

No.	Between Stations.	Observed Angle.	Weight.
1	Big Rock and Small Rock	99° 42′ 15″.61	137
2	Small Rock and Tokus	133 39 05.07	22
3	Small Rock and Buzzard	40 12 52.43	57
4	Buzzard and Tokus	93 26 13.14	50
5	Tokus and Big Rock	126 38 40.69	20

Ans. 99° 42′ 15″.46, etc.

4. The following observations of the temperature at different depths were taken at the boring of the deep artesian well at Grenelle in France, the mean yearly temperature at the surface being $10^\circ.60$ centigrade:

1. Temperature at a depth of 28 meters = 11.71 degrees.
2. Temperature at a depth of 66 meters = 12.90 degrees.
3. Temperature at a depth of 173 meters = 16.40 degrees.
4. Temperature at a depth of 248 meters = 20.00 degrees.
5. Temperature at a depth of 298 meters = 22.20 degrees.
6. Temperature at a depth of 400 meters = 23.75 degrees.
7. Temperature at a depth of 505 meters = 26.45 degrees.
8. Temperature at a depth of 548 meters = 27.70 degrees.

Deduce from these observations the empirical formula

$$t = 10^\circ.6 + 0.0415x - 0.0000193x^2,$$

where t is the temperature at a depth of x meters.

5. Gordon's formula for the ultimate strength of columns may be written

$$c = \frac{S}{1 + Tj^2},$$

in which c is the crushing-load per unit of area of cross-section, j the ratio of the length of the column to its least diameter, and S and T are constants to be found by experiment. Determine the best values of these constants for the following four experiments on wrought-iron Phœnix columns:

$$\begin{array}{lllll} c = & 34650, & 35000, & 36580, & 37030. \\ j = & 42, & 33, & 24, & 19.5. \end{array}$$

6. From several census records of the population of the United States deduce an empirical formula showing the population for any year.

CHAPTER VIII.

CONDITIONED OBSERVATIONS.

113. The general method of adjusting conditioned observations has been deduced in Arts. 56, 57, and that of investigating the precision in Arts. 77, 78. The following is the process:

1st, Having given n observations upon q quantities subject to n' rigorous conditions, the first step is to represent the quantities by symbols, and state n observation equations and n' conditional equations. Generally it will be found most convenient to take the unknown quantities as representing corrections to assumed approximate values, and to state the observation and conditional equations in terms of these corrections.

2d, From the n' conditional equations find the values of n' unknown quantities in terms of the remaining $q - n'$ quantities, and substitute these values in the n observation equations, each of which then represents an independent observation.

3d, Adjust these n observation equations by the method of Chap. VII, and find the most probable values of the $q - n'$ quantities. Then, by substitution in the conditional equations, the most probable values of the remaining n' quantities are known.

4th, Insert the adjusted values in the n observation equations, and find the residuals, and then, from (33), the probable error of an observation of the weight unity. If desired, the weights of the adjusted values may be found by Art. 75, and their probable errors by (31).

114. The special method of correlatives, which is particularly valuable in the adjustment of geodetic triangulations, has been explained in Art. 58. In order to apply it, the local adjustments should first be made; so that for each quantity, $z_1, z_2 \ldots z_q$, a value, $M_1, M_2 \ldots M_q$, called the observed value, is known. The numbers q and n are hence equal. The following are the steps of the practical application:

1st, For the rigorous conditions write n' conditional equations, as in (14). Substitute in these the observed values, M_1, $M_2 \ldots M_q$, for the quantities $z_1, z_2 \ldots z_q$; and let $d_1, d_2 \ldots d_q$ be the differences or discrepancies that arise.

2d, Assume n' new unknown quantities, or correlatives, $K_1, K_2 \ldots K_{n'}$, and write the normal equations (16). Solve these normal equations, and thus find the values of the correlatives.

3d, From (15) find the corrections $v_1, v_2 \ldots v_q$, which, when applied to the observed values $M_1, M_2 \ldots M_q$, give the most probable adjusted values.

4th, Compute the sum Σpv^2, and from (34) find the probable error of an observation of the weight unity. The probable error of any observed M is then easily found from (31), and that of the corresponding adjusted value is somewhat smaller, since the weights are increased by the adjustment.

Angles of a Triangle.

115. When the three observed angles of a plane triangle are of equal weight, it is easy to show that the correction to be applied to each is one-third of the discrepancy between their sum and 180°. The following is the proof by the method of correlatives:

1st, Let M_1, M_2, and M_3 be the observed values, and z_1, z_2,

and z_3 the required most probable values. The conditional equation is

$$z_1 + z_2 + z_3 - 180° = 0.$$

Substitute in this the observed values, and it does not reduce to zero, but leaves a small discrepancy d; thus

$$M_1 + M_2 + M_3 - 180° = d.$$

By comparison with (14) it is seen that $\alpha_1 = \alpha_2 = \alpha_3 = + 1$.

2d, Take K as the single correlative. The weights are all equal, or $p = 1$. From (16) the single normal equation is

$$[\alpha\alpha].K + d = 0, \quad \text{or} \quad 3K + d = 0,$$

from which $K = -\frac{1}{3}d$.

3d, From (15) the three corrections now are

$$v_1 = -\frac{d}{3}, \qquad v_2 = -\frac{d}{3}, \qquad v_3 = -\frac{d}{3},$$

and, accordingly, the most probable values of the three angles are

$$z_1 = M_1 - \frac{d}{3}, \qquad z_2 = M_2 - \frac{d}{3}, \qquad z_3 = M_3 - \frac{d}{3}.$$

4th, The sum of the squares of the residuals is $\frac{d^2}{3}$, and hence by (34) the probable error of a single observed angle is $0.39d$. By working the problem according to the general method of Art. 113, it may be shown (as in Art. 103) that the probable error of an adjusted angle is $0.32d$.

116. When the three observed angles of a plane triangle are of unequal weights, it is easy to show that the corrections to be

applied are inversely as the weights. For instance, take the following numerical case:

$$
\begin{array}{rlll}
M_1 = & 36^\circ\ 25'\ 47'', & \text{with weight } 4 \\
M_2 = & 90\ \ 36\ \ 28, & \text{with weight } 2 \\
M_3 = & 52\ \ 57\ \ 57, & \text{with weight } 3 \\
\hline
\text{Sum} = & 180^\circ\ 00'\ 12'' &
\end{array}
$$

1st, Take z_1, z_2, and z_3 as the most probable values; then, as before, the conditional equation is

$$z_1 + z_2 + z_3 - 180^\circ = 0.$$

The discrepancy is $12''$. To compare with (14), (15), and (16), $a_1 = a_2 = a_3 = +1$, $p_1 = 4$, $p_2 = 2$, and $p_3 = 3$.

2d, Only one correlative is necessary; and from (16) the single normal equation is

$$(\tfrac{1}{4} + \tfrac{1}{2} + \tfrac{1}{3})K + 12 = 0,$$

and hence $K = -\frac{144}{13} = -11.08$.

3d, From (15) the corrections now are

$$v_1 = \frac{K}{4} = -2''.77, \qquad v_2 = -5''.54, \qquad v_3 = -3''.69,$$

and the adjusted angles are

$$
\begin{array}{rlll}
z_1 = & 36^\circ & 25' & 44''.23 \\
z_2 = & 90 & 36 & 22.46 \\
z_3 = & 52 & 57 & 53.31 \\
\hline
\text{Sum} = & 180^\circ & 00' & 00''.00
\end{array}
$$

4th, The residuals are the three corrections v_1, v_2, and v_3, and the sum of their weighted squares is $\Sigma pv^2 = 132.92$, from which, by (34), $r = 7''.77$ for the probable error of an observa-

tion of the weight unity. By (31) the probable errors of the observed values are found to be

$$r_1 = 3''.89, \qquad r_2 = 5''.50, \qquad r_3 = 4''.49,$$

and the probable errors of the adjusted values are somewhat less than these.

The adjustment of the angles of a spherical triangle differs from that of a plane triangle only in the introduction of the spherical excess into the conditional equation; thus $s + t + u = 180° +$ spherical excess.

Angles at a Station.

117. When n angles, and also their sum, are observed at a station, and the weights are all equal, it is easy to show, as in Art. 103, that the correction to be applied to each observed angle is $\frac{1}{n+1}$th of the discrepancy between the observed sum and the sum of the observed single angles.

When n angles, which close the horizon, are observed at a station, and the weights are equal, it is easy to show, as in Art. 115, that the correction to be applied to each observed angle is $\frac{1}{n}$th of the discrepancy between 360° and the sum of the observed angles.

When angles at a station close the horizon, or are observed by sums or differences, the adjustment may be effected, either for equal or unequal weights, by the method of Art. 113, or by that of Art. 114. The former will always reduce to the method of independent observations, as exemplified in Arts. 103–105.

118. As an example of the application of the method of correlatives, consider the observations of Art. 104. Represent the most probable values of the seven angles by $z_1, z_2 \ldots z_7$.

From Fig. 7 the following conditions are seen:

$$
\begin{aligned}
z_1 - z_2 + z_3 \qquad\qquad &= 0, \\
z_4 - z_5 + z_6 + z_7 \qquad\qquad &= 0, \\
z_1 + z_3 + z_4 + z_6 + z_7 - 360^\circ &= 0.
\end{aligned}
$$

By substituting in these the observed values, the following discrepancies are found:—

$$d_1 = -0.210, \qquad d_2 = -0.648, \qquad d_3 = -0.420.$$

Take K_1, K_2, and K_3 as the correlatives to be determined. By comparison with (14), it is seen that

$$
\begin{aligned}
&\alpha_1 = +1, \quad \alpha_2 = -1, \quad \alpha_3 = +1, \quad \alpha_4 = \alpha_5 = \alpha_6 = \alpha_7 = 0, \\
&\beta_1 = \beta_2 = \beta_3 = 0, \quad \beta_4 = \beta_6 = \beta_7 = +1, \quad \beta_5 = -1, \\
&\gamma_1 = \gamma_3 = \gamma_4 = \gamma_6 = \gamma_7 = +1, \quad \gamma_2 = \gamma_5 = 0.
\end{aligned}
$$

All weights are unity. The three normal equations then are, from (16),

$$
\begin{aligned}
3K_1 \qquad\quad + 2K_3 - 0.210 &= 0, \\
+ 4K_2 + 3K_3 - 0.648 &= 0, \\
2K_1 + 3K_2 + 5K_3 - 0.420 &= 0,
\end{aligned}
$$

and their solution gives

$$K_1 = +0.167, \qquad K_2 = +0.271, \qquad K_3 = -0.145.$$

From (15) the corrections now are

$$
\begin{aligned}
v_1 &= +K_1 + K_3 = +0''.022, \\
v_2 &= -K_1 \qquad\quad = -\ 0.167, \\
v_3 &= +K_1 + K_3 = +\ 0.022, \\
v_4 &= +K_2 + K_3 = +\ 0.126, \\
v_5 &= +K_2 \qquad\quad = +\ 0.271, \\
v_6 &= +K_2 + K_3 = +\ 0.126, \\
v_7 &= +K_2 + K_3 = +\ 0.126,
\end{aligned}
$$

and if these be applied to the observed values M_1, M_2 . . . M_7, the adjusted values are found the same, within one or two thousandths of a second, as in Art. 103, the slight difference being due to the neglect of the fourth decimal places.

Angles of a Quadrilateral.

119. In a quadrilateral $WXYZ$, the two single angles at each corner are equally well measured. It is required to adjust them, so that the sum of the three angles in each triangle shall equal 180°, and the sum of the four angles of the quadrilateral shall equal 360°.

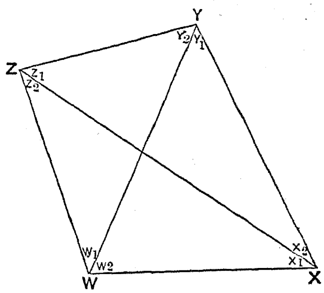


Fig. 10.

Let the measured angles at the corner W be denoted by W_1 and W_2, and similarly for each of the other corners, as shown in Fig. 10. Let w_1 and w_2 be corrections to be applied to W_1 and W_2 in order to give the most probable values, $W_1 + w_1$ and $W_2 + w_2$.

In order to avoid writing identical equations, select any corner, as W, and take the three triangles, WXZ, ZWY, and XYW which meet at that point, as the three triangles for correction. Evidently, if the angles of these triangles add up to 180°, those of the fourth triangle will also. The three conditional equations now are

$$
\begin{aligned}
w_1 + w_2 + x_1 + z_2 + d_1 &= 0, \\
w_2 + x_1 + x_2 + y_1 + d_2 &= 0, \\
w_1 + y_2 + z_1 + z_2 + d_3 &= 0,
\end{aligned}
$$

in which d_1, d_2, and d_3 denote the differences or discrepancies

between the sum of the measured angles of the triangles and the theoretic sum 180°; thus, for example,

$$W_1 + W_2 + X_1 + Z_2 - 180° = d_1.$$

From the three conditional equations the values of the eight corrections are to be found, either by the method of Art. 113 or by that of Art. 114. The latter will be the shorter. Assume, then, three correlatives, K_1, K_2, and K_3, and for each correction write a correlative equation, thus

$$\begin{aligned}
+K_1 + K_3 &= w_1,\\
+K_1 + K_2 &= w_2,\\
+K_1 + K_2 &= x_1,\\
+K_1 + K_3 &= z_2,\\
+K_2 &= x_2,\\
+K_2 &= y_1,\\
+K_3 &= y_2,\\
+K_3 &= z_1,
\end{aligned}$$

the co-efficients of K_1 being the co-efficients of the corresponding unknown quantities in the first conditional equation, and so on. From these equations the three normal equations are

$$\begin{aligned}
4K_1 + 2K_2 + 2K_3 + d_1 &= 0,\\
2K_1 + 4K_2 + d_2 &= 0,\\
2K_1 + 4K_3 + d_3 &= 0,
\end{aligned}$$

whose solution gives the values of K_1, K_2, and K_3; and, inserting these in the correlative equations, the following values of the corrections are found:

$$\begin{aligned}
w_1 = z_2 &= \tfrac{1}{8}(-2d_1 + d_2 - d_3),\\
w_2 = x_1 &= \tfrac{1}{8}(-2d_1 - d_2 + d_3),\\
x_2 = y_1 &= \tfrac{1}{8}(2d_1 - 3d_2 - d_3),\\
y_2 = z_1 &= \tfrac{1}{8}(2d_1 - d_2 - 3d_3),
\end{aligned}$$

and the addition of these to the observed angles gives the adjusted values.

For example, let the following angles be given:

$$\begin{array}{llll} W_1 = 41^\circ & 58' & 47'', \\ W_2 = 64 & 08 & 34, \\ X_1 = 36 & 34 & 15, \\ X_2 = 29 & 59 & 51, \end{array} \qquad \begin{array}{llll} Y_1 = 49^\circ & 17' & 30'', \\ Y_2 = 53 & 53 & 51, \\ Z_1 = 46 & 49 & 16, \\ Z_2 = 37 & 18 & 18. \end{array}$$

Here the discrepancies are

$$\begin{aligned} d_1 &= W_1 + W_2 + X_1 + Z_2 - 180^\circ = -6'', \\ d_2 &= W_2 + X_1 + X_2 + Y_1 - 180 \ = +10, \\ d_3 &= W_1 + Y_2 + Z_1 + Z_2 - 180 \ = +12. \end{aligned}$$

Then, by the above formulas, the corrections are

$$\begin{array}{ll} w_1 = z_2 = +1''.25, & w_2 = x_1 = +1''.75, \\ x_2 = y_1 = -\ 6.75, & y_2 = z_1 = -\ 7.25, \end{array}$$

so that the adjusted values are

$$\begin{array}{llll} W_1 + w_1 = 41^\circ & 58' & 48''.25, \\ W_2 + w_2 = 64 & 08 & 35.75, \\ X_1 + x_1 = 36 & 34 & 16.75, \\ X_2 + x_2 = 29 & 59 & 44.25, \end{array} \qquad \begin{array}{llll} Y_1 + y_1 = 49^\circ & 17' & 23''.25, \\ Y_2 + y_2 = 53 & 53 & 43.75, \\ Z_1 + z_1 = 46 & 49 & 08.75, \\ Z_2 + z_2 = 37 & 18 & 19.25. \end{array}$$

These angles now fulfil all the geometrical conditions required in the statement of the problem, and are, furthermore, the most probable angles.

120. If the large angles at the corners are measured as well as the single angles, the most convenient method of procedure is, first to make the station adjustment at each corner, and then, with the eight single angles, to make a further adjustment, as in the last article. The following is an example illustrating the

steps of the process for the case of unequal weights. Let the twelve measured angles be

$W = 106°\ 07'\ 27''$, weight 3,	$Y = 103°\ 11'\ 15''$, weight 2,
$W_1 = 41\ 58\ 47$, weight 3,	$Y_1 = 49\ 17\ 30$, weight 2,
$W_2 = 64\ 08\ 34$, weight 3,	$Y_2 = 53\ 53\ 51$, weight 2,
$X = 66\ 34\ 03$, weight 1,	$Z = 84\ 07\ 30$, weight 4,
$X_1 = 36\ 34\ 21$, weight 1,	$Z_1 = 46\ 49\ 16$, weight 1,
$X_2 = 29\ 59\ 45$, weight 1,	$Z_2 = 37\ 18\ 18$, weight 2.

First, by Art. 117, make the station adjustment at each corner, and obtain the following results:

$W_1 = 41°\ 58'\ 49''.0$, weight $\frac{9}{2}$,	$Y_1 = 49°\ 17'\ 28''.0$, weight 3,
$W_2 = 64\ 08\ 36.0$, weight $\frac{9}{2}$,	$Y_2 = 53\ 53\ 49.0$, weight 3,
$X_1 = 36\ 34\ 20.0$, weight $\frac{3}{2}$,	$Z_1 = 46\ 49\ 13.7$, weight $\frac{7}{3}$,
$X_2 = 29\ 59\ 44.0$, weight $\frac{3}{2}$,	$Z_2 = 37\ 18\ 16.9$, weight $\frac{14}{5}$.

Next let w_1, w_2, etc., be corrections to these values in order to satisfy the geometrical requirements of the figure. Then, as in the preceding article, the three conditional equations are

$$\begin{aligned} w_1 + w_2 + x_1 + z_2 + 1''.9 &= 0, \\ w_2 + x_1 + x_2 + y_1 + 8.0 &= 0, \\ w_1 + y_2 + z_1 + z_2 + 8.6 &= 0. \end{aligned}$$

From (15) the eight correlative equations are

$$\begin{aligned} w_1 &= \tfrac{2}{9}(K_1 \qquad\quad + K_3), \\ w_2 &= \tfrac{2}{9}(K_1 + K_2 \qquad\ ), \\ x_1 &= \tfrac{2}{3}(K_1 + K_2 \qquad\ ), \\ x_2 &= \tfrac{2}{3}(\qquad + K_2 \qquad\ ), \\ y_1 &= \tfrac{1}{3}(\qquad + K_2 \qquad\ ), \\ y_2 &= \tfrac{1}{3}(\qquad\qquad\quad + K_3), \\ z_1 &= \tfrac{3}{7}(\qquad\qquad\quad + K_3), \\ z_2 &= \tfrac{5}{14}(K_1 \qquad\quad + K_3). \end{aligned}$$

From (16) the three normal equations now are

$$\begin{aligned}
(\tfrac{2}{9}+\tfrac{2}{9}+\tfrac{2}{3}+\tfrac{5}{14})K_1 + \qquad (\tfrac{2}{9}+\tfrac{2}{3})K_2 + (\tfrac{2}{9}+\tfrac{5}{14})K_3 + 1.9 &= 0,\\
(\tfrac{2}{9}+\tfrac{2}{3})K_1 + (\tfrac{2}{9}+\tfrac{2}{3}+\tfrac{2}{3}+\tfrac{1}{3})K_2 \qquad\qquad + 8.0 &= 0,\\
(\tfrac{2}{9}+\tfrac{5}{14})K_1 \qquad\qquad + (\tfrac{2}{9}+\tfrac{1}{3}+\tfrac{3}{7}+\tfrac{5}{14})K_3 + 8.6 &= 0.
\end{aligned}$$

and their solution gives the values

$$K_1 = +6.99, \quad K_2 = -7.53, \quad K_3 = -9.43,$$

from which the following corrections are found:

$$\begin{aligned}
w_1 &= -0''.5, & x_1 &= -0.4, & y_1 &= -2.5, & z_1 &= -4.1,\\
w_2 &= -\ 0.1, & x_2 &= -5.0, & y_2 &= -3.1, & z_2 &= -0.9.
\end{aligned}$$

The final adjusted values of the single angles now are

$$\begin{aligned}
W_1 &= 41^\circ\ 58'\ 48''.5, & Y_1 &= 49^\circ\ 17'\ 25''.5,\\
W_2 &= 64\ \ 08\ \ 35.9, & Y_2 &= 53\ \ 53\ \ 45.9,\\
X_1 &= 36\ \ 34\ \ 19.6, & Z_1 &= 46\ \ 49\ \ 09.6,\\
X_2 &= 29\ \ 59\ \ 39.0, & Z_2 &= 37\ \ 18\ \ 16.0.
\end{aligned}$$

The adjusted values of the large angles are now obtained by simple addition of the single angles, and are

$$\begin{aligned}
W &= 106^\circ\ 07'\ 24''.4, & Y &= 103^\circ\ 11'\ 11''.4,\\
X &= 66\ \ 33\ \ 58.6, & Z &= 84\ \ 07\ \ 25.6,
\end{aligned}$$

whose sum is exactly 360 degrees.

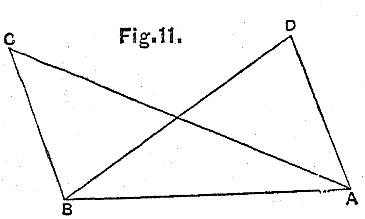
121. In geodetic surveys where the sides of the quadrilateral are many miles in length, the spherical excess must be considered in stating the conditional equations for the three triangles. In such work a fourth conditional equation must also be introduced in order to insure that the length of any side shall be the same through whatever set of triangles it be computed. The development of the calculations for such cases belongs properly to works on geodesy, and will not here be discussed. Detailed examples of the method may be seen in

Schott's article on the adjustment of the horizontal angles of a triangulation in the United States Coast Survey Report for 1854, in Clarke's Geodesy (Oxford, 1880), and in many German works on higher surveying.*

Simple Triangulation.

122. In the adjustment of a simple triangulation the method of procedure is essentially the same as for a quadrilateral. First, the adjustment of the angles at each station should be made, and then the resulting values further corrected, so as to satisfy the geometrical requirements of the figure. This method is not strictly in accordance with the fundamental principle of Least Squares. By the station adjustment a correction, v_1, is found for each angle, and by the figure adjustment another correction, v_2; so that the total correction is $v_1 + v_2$. The fundamental principle for observations of equal weight requires that $\Sigma(v_1 + v_2)^2$ should be made a minimum in order to obtain the best values of the corrections, while by the method pursued Σv_1^2 is made a minimum in the first adjustment, and Σv_2^2 a minimum in the second. The reason for deviating from the strict letter of the law is, that the general method of determining the total equation at once is too laborious, owing to the large number of conditional equations involved. Usually also the difference between the final results of the two methods will be small. In the next article will be given a comparison of the two methods as applied to a simple case.

Fig. 11.



* See also Merriman's Elements of Precise Surveying and Geodesy. New York, 1899.

123. The following observations were made to determine the distance between the non-intervisible stations C and D by means of a measured base AB:

$$\begin{aligned}
BAC &= 27^\circ\ 09'\ 05''.5,\\
BAD &= 51\ \ 34\ \ 35.5,\\
CAD &= 24\ \ 25\ \ 27.8,\\
ABD &= 70\ \ 08\ \ 32.1,\\
ABC &= 128\ \ 29\ \ 07.5,\\
DBC &= 58\ \ 20\ \ 38.4,\\
ACB &= 24\ \ 21\ \ 46.0,\\
ADB &= 58\ \ 16\ \ 50.8.
\end{aligned}$$

By the strict method of Art. 113 or Art. 114 the four conditional observations are written, one for each of the points A and B, and one for each triangle, and the adjusted values found as given in the second column of the following table:

Observed.	Adjusted.	v.	v^2.
05″.5	06.2	+ 0.7	0.49
35.5	35.6	+ 0.1	0.01
27.8	29.4	+ 1.6	2.56
32.1	32.0	− 0.1	0.01
07.5	08.6	+ 1.1	1.21
38.4	36.6	− 1.8	3.24
46.0	45.2	− 0.8	0.64
50.8	52.4	+ 1.6	2.56

The sum Σv^2 is here 10.72, and by (34) the probable error of a single observation is

$$r = 0.6745\sqrt{\frac{10.72}{4}} = 1''.1.$$

By the shorter method the local adjustment at A and B is first made, giving the results

$$
\begin{aligned}
BAC &= 27^\circ\ 09'\ 06''.2, && \text{weight } 1.5,\\
BAD &= 51\ \ 34\ \ 34.8, && \text{weight } 1.5,\\
ABD &= 70\ \ 08\ \ 31.1, && \text{weight } 1.5,\\
ABC &= 128\ \ 29\ \ 08.5, && \text{weight } 1.5.
\end{aligned}
$$

The triangles ABC and BAD are next separately adjusted, using these four angles and those at C and D. The results are

Observed.	Adjusted.	v.	v^2.
05″.5	06.0	+ 0.5	0.25
35.5	35.7	+ 0.2	0.04
27.8	29.7	+ 1.9	3.61
32.1	32.0	− 0.1	0.01
07.5	08.3	+ 0.8	0.64
38.4	36.3	− 2.1	4.41
46.0	45.7	− 0.3	0.09
50.8	52.3	+ 1.5	2.25

The sum Σv^2 is here 11.3, which is but slightly greater than that of the stricter method. A comparison of the two sets of adjusted values shows also that the differences are small.

Levelling.

124. A simple discussion of the precision of levelling observations involving but one conditional equation will here be given as an illustration of the general method of treatment of Art. 113.

There are three points, A, B, and C, situated at nearly equal distances apart, but upon different levels. In order to ascertain

with accuracy their relative heights, a levelling instrument was set up between A and B, and readings taken upon a rod held at those points, with the results,

On rod at A, 8.7342 feet, mean of 12 readings.
On rod at B, 2.3671 feet, mean of 9 readings.

The instrument was then moved to a point between B and C, and the observations taken.

On rod at B, 5.0247 feet, mean of 7 readings,
On rod at C, 11.2069 feet, mean of 4 readings.

Lastly, the level was set up between C and A, and the rods observed.

On rod at C, 0.4672 feet, mean of 5 readings,
On rod at A, 0.6510 feet, mean of 3 readings.

It is required to find the adjusted values of these readings, the most probable differences of level between the points, and the probable error of a single reading on the rod.

First arrange these measurements as they would be written in an engineer's level-book, and, assuming the elevation of A as 0.0, find the heights of the other points.

Station.	Back Sight.	Fore Sight.	Height of Instrument.	Elevation above A.
${}_{12}A$	8.7342		8.7342	0.0
${}_{7}B_{9}$	5.0247	2.3671	11.3918	6.3671
${}_{5}C_{4}$	0.4672	11.2069	0.6521	0.1849
A_3		0.6510		0.0011

The number of readings or the weight of each sight is placed in the first column preceding and following the name of the

station; thus ${}_7B_9$ denotes that the back sight on B has a weight of 7, and the fore sight one of 9. Regarding the elevation of A as 0, that of B comes out 6.3671 feet, that of C, 0.1849 feet; and, on returning to the starting-point, it is found that A is 0.0011 feet, instead of 0 as it ought to be.

Represent the back sights upon A, B, and C by Z_1, Z_3, and Z_5, and the fore sights upon B, C, and A by Z_2, Z_4, and Z_6, and let z_1, z_3, z_5, z_2, z_4, and z_6 be corrections to be applied to those observed values. The observation equations then are

$$\begin{aligned} z_1 &= 0, \text{ weight } 12, & z_2 &= 0, \text{ weight } 9, \\ z_3 &= 0, \text{ weight } 7, & z_4 &= 0, \text{ weight } 4, \\ z_5 &= 0, \text{ weight } 5, & z_6 &= 0, \text{ weight } 3, \end{aligned}$$

and the conditional equation is

$$z_1 + z_3 + z_5 - z_2 - z_4 - z_6 = -0.0011.$$

From the conditional equation take the value of z_4, and insert it in the observation equations, which, after multiplication by the square roots of their respective weights, become

$$\begin{aligned} \sqrt{12}\, z_1 &= 0, \\ \sqrt{7}\, z_3 &= 0, \\ \sqrt{5}\, z_5 &= 0, \\ 3 z_2 &= 0, \\ \sqrt{3}\, z_6 &= 0, \\ 2z_1 + 2z_3 + 2z_5 - 2z_2 - 2z_6 &= -0.0022. \end{aligned}$$

From these the normal equations (Art. 48) are

$$\begin{aligned} 16z_1 + 4z_3 + 4z_5 - 4z_2 - 4z_6 &= -0.0044, \\ 4z_1 + 11z_3 + 4z_5 - 4z_2 - 4z_6 &= -0.0044, \\ 4z_1 + 4z_3 + 9z_5 - 4z_2 - 4z_6 &= -0.0044, \\ -4z_1 - 4z_3 - 4z_5 + 13z_2 + 4z_6 &= +0.0044, \\ -4z_1 - 4z_3 - 4z_5 + 4z_2 + 7z_6 &= +0.0044, \end{aligned}$$

the first being the normal equation for z_1, the second for z_3, the third for z_5, the fourth for z_2, and the fifth for z_6. The solution gives the following results:

$$z_1 = -0.00008, \quad z_3 = -0.00014, \quad z_5 = -0.00020,$$
$$z_2 = +0.00011, \quad z_4 = +0.00024, \quad z_6 = +0.00033.$$

Applying these to the observed values, the adjusted results are

Station.	Back Sight.	Fore Sight.	Elevation above *A*.
A	8.73412		0.0
B	5.02456	2.36721	6.36691
C	0.46700	11.20714	0.18433
A		0.65133	0.0

The residuals are in this case the corrections z_1, z_3, etc. Squaring these, multiplying each square by its weight, and adding, gives

$$\Sigma pv^2 = 0.000001079.$$

From formula (34) then

$$r = 0.6745\sqrt{0.000001079} = 0.0007 \text{ feet},$$

which is the probable error of a single reading on the rod.

125. The adjustment of a network of level lines may also be effected by the method of conditioned observations. When the levelling is of the same precision throughout, the probable errors of differences of level should be taken as varying with the square root of the lengths of lines, being governed, in short,

by the same law of propagation of error as linear measurements (see Art. 91). Each difference of level should hence be assigned a weight inversely proportional to the length of the line between the two points. For each triangle or polygon of the network, there is the rigorous condition that the sum of the differences of level shall be zero. From these conditional equations, corrections to the observed differences of level are determined by the method of Art. 114.

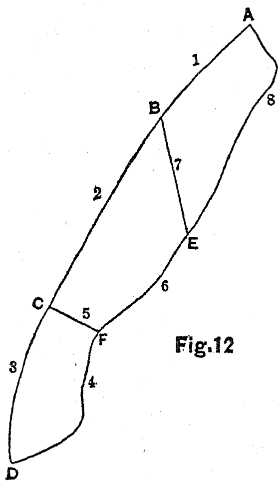


Fig. 12

As an example, consider the following eight differences of level forming three closed figures, *ABE*, *BCFE*, and *CDF*:

No.	Stations.	Diff. Level.	Distance.	Weight.
		Feet.	Miles.	
1	*B* above *A*	120.2	4.0	0.25
2	*C* above *B*	230.6	7.2	0.14
3	*D* above *C*	143.0	5.0	0.20
4	*D* above *F*	294.4	6.3	0.16
5	*C* above *F*	150.2	2.0	0.50
6	*F* above *E*	93.4	4.8	0.21
7	*B* above *E*	14.5	3.5	0.29
8	*E* above *A*	106.7	8.3	0.12

It is required to find the most probable corrections to the above differences of level in order to cause the discrepancies in the three polygons to vanish.

Let h_1, h_2, etc., represent the most probable differences of level. Then the three conditions are

$$\begin{array}{ll} \text{for } ABE, & h_1 - h_7 - h_8 = 0, \\ \text{for } BCFE, & h_2 - h_5 - h_6 + h_7 = 0, \\ \text{for } CDF, & h_3 - h_4 + h_5 = 0. \end{array}$$

Let v_1, v_2, etc., be the most probable corrections to the observed differences of level, so that

$$h_1 = 120.2 + v_1, \qquad h_2 = 230.6 + v_2, \qquad \text{etc.}$$

Then the three conditional equations become

$$\begin{aligned} v_1 - v_7 - v_8 - 1.0 &= 0, \\ v_2 - v_5 - v_6 + v_7 + 1.5 &= 0, \\ v_3 - v_4 + v_5 - 1.2 &= 0. \end{aligned}$$

From these the correlative equations are written, the weight of each v being taken as the reciprocal of the corresponding distance:

$$\begin{aligned} v_1 &= +\,4.0K_1, \\ v_2 &= +\,7.2K_2, \\ v_3 &= +\,5.0K_3, \\ v_4 &= -\,6.3K_3, \\ v_5 &= -\,2.0K_2 + 2.0K_3, \\ v_6 &= -\,4.8K_2, \\ v_7 &= -\,3.5K_1 + 3.5K_2, \\ v_8 &= -\,8.3K_1. \end{aligned}$$

Next the three normal equations are

$$\begin{aligned} 15.8K_1 - 3.5K_2 \qquad\qquad - 1.0 &= 0, \\ -\,3.5K_1 + 17.5K_2 - 2.0K_3 + 1.5 &= 0, \\ -\,2.0K_2 + 13.3K_3 - 1.2 &= 0, \end{aligned}$$

and the solution of these gives

$$K_1 = +\,0.04848, \qquad K_2 = -\,0.066855, \qquad K_3 = +\,0.08017$$

Lastly, by substituting these in the correlative equations, the corrections are found, which are given in the third column of the following table, while in the fourth are the adjusted results.

No.	Observed Diff. Level.	v.	Adjusted Diff. Level.
1	120.2	+ 0.19	120.39
2	230.6	− 0.48	230.12
3	143.0	+ 0.40	143.40
4	294.4	− 0.51	293.89
5	150.2	+ 0.29	150.49
6	93.4	+ 0.32	93.72
7	14.5	− 0.40	14.10
8	106.7	− 0.40	106.30

In order to ascertain the precision of the work, the corrections are squared, and each square multiplied by its respective weight, and the sums of these products taken. This sum is about 0.246; and then by (34) the probable error of an observation of the weight unity, that is, the probable error of the difference of level of the ends of a line one mile in length, is

$$r = 0.6745\sqrt{\frac{0.246}{3}} = 0.19 \text{ feet},$$

a result that indicates a low degree of precision.

126. *Problems.*

1. Adjust the following angles taken at the station O:

$AOB = 40° \ 52' \ 37''$, weight 16,
$BOC = 92 \ \ 25 \ \ 41$, weight 4,
$COD = 80 \ \ 6 \ \ 15$, weight 3,
$DOA = 146 \ \ 35 \ \ 20$, weight 1.

2. In a spherical triangle XYZ the three measured angles are

$$\begin{aligned} X &= 93^\circ\ 48'\ 15''.22, && \text{with weight } 30, \\ Y &= 51\ \ \ 55\ \ \ \ 0.18, && \text{with weight } 19, \\ Z &= 34\ \ \ 16\ \ \ 49.72, && \text{with weight } 13. \end{aligned}$$

The spherical excess is $4''.05$. What are the adjusted angles?

3. In a quadrilateral $WXYZ$, the following angles, all of equal weight, are measured, and it is required to adjust them.

$$\begin{aligned} W &= 106^\circ\ 07'\ 30'', & Y_1 &= 49^\circ\ 17'\ 23'', \\ W_1 &= \ \ 41\ \ \ 58\ \ \ 47, & Y_2 &= 53\ \ \ 53\ \ \ 50, \\ W_2 &= \ \ 64\ \ \ 08\ \ \ 34, & Z &= 84\ \ \ 07\ \ \ 18, \\ X &= \ \ 66\ \ \ 34\ \ \ 09, & Z_2 &= 37\ \ \ 18\ \ \ 12. \\ X_1 &= \ \ 36\ \ \ 34\ \ \ 21, \end{aligned}$$

4. Adjust the level observations in Art. 100 by the method of conditioned observations, taking the weights as equal.

5. Discuss the method of correcting the latitudes and departures in a compass survey of a field.

6. Two bases, AB and DE, are connected by three triangles, ABC, BCD, and CDE. The bases are measured, and also the three angles of each triangle. State the four conditional equations, and explain in detail the process of adjustment.

CHAPTER IX.

THE DISCUSSION OF OBSERVATIONS.

127. In the preceding pages it has been shown how to adjust observations, and how to ascertain their precision by means of the probable error. By thus treating series or sets of measurements, a comparison or discussion may be instituted concerning the relative degrees of precision, the presence of constant errors, and the best way to improve the methods of observation. In this chapter it is proposed to present some further remarks relating to the discussion of observations by the use of the fundamental law of probability of error, and to indicate that this law is also applicable to social statistics, and that it really governs the way in which the laws of nature are executed.

Probability of Errors.

128. In Chap. II a method of investigating the probability of errors, and comparing theory with experience, was given, in which it was necessary to assume the value of the measure of precision h. For instance, in Arts. 19 and 33 there are discussed one hundred residual errors, for which the value of h is stated to be $\frac{1}{2''.24}$. It is now easy to see that this value may be found at once from the probable error r by means of the formula (17), while r is deduced from the formula (20). To

compare, then, the theoretical and actual distribution of errors for such cases by the use of Table I it is only necessary to deduce the value of r in the usual way, and from it to find h, which enters as an argument in the table.

It is evident, then, that, in undertaking such discussions, it is more convenient to have a table of the values of the probability integral in terms of r as an argument. Such is Table II at the end of this book, which gives, for successive values of $\frac{x}{r}$, the probability that a given error is less numerically than x, or that it lies between the limits $-x$ and $+x$.

To illustrate the use of Table II consider an angle for which the mean value is found to be

$$37^\circ \quad 42' \quad 13''.92 \pm 0''.25.$$

Now, from the definition of probable error, it is known that the probability is $\frac{1}{2}$ that the actual error of the result is less than $0''.25$. Let it be asked what are the respective probabilities that the actual error is less than the amounts $0''.5$ and $1''.0$. From the table

$$\text{for } \frac{x}{r} = \frac{0.50}{0.25} = 2, \qquad P = 0.823,$$

$$\text{for } \frac{x}{r} = \frac{1.00}{0.25} = 4, \qquad P = 0.993.$$

Hence the probability that the error in the result is less than $0''.5$ is 0.823, or it is a fair wager of 823 to 177 that such is the case. And the probability that the error is less than $1''.0$ is 0.993, or it is a fair wager of 993 to 7 that such is the case.

As the number of errors is proportional to the probability, the values of the integral need only to be multiplied by the total number of errors to give the theoretical number less than

certain limits. For example, in one thousand errors or residuals, there should be

264 less than $\frac{1}{2}r$,	and	736	greater,
500 less than r,	and	500	greater,
823 less than $2r$,	and	177	greater,
957 less than $3r$,	and	43	greater,
993 less than $4r$,	and	7	greater,
999 less than $5r$,	and	1	greater.

Table II gives only four decimal places, which suffice for any ordinary investigation. By the methods of calculation explained in Chap. II more decimals may be deduced, and the following results be found for the theoretical distribution of errors when the total number of errors is one hundred thousand:

95698 are less than $3r$,	and	4302	greater,
99302 are less than $4r$,	and	698	greater,
99926 are less than $5r$,	and	74	greater,
99995 are less than $6r$,	and	5	greater.

As the frequency with which an error occurs is expressed by its probability, it is evident that errors greater than five or six times the probable error should be very rare.

129. As shown in Art. 35, the probability of the error 0 is $\frac{h.dx}{\sqrt{\pi}}$, or, introducing for h its value $\frac{0.4769}{r}$, it may be written

$$y_0 = 0.2691 \frac{dx}{r}.$$

Here dx is the interval between successive values of x. If there be N errors in a series, the number having the value 0 should hence be

$$N_0 = 0.2691 \frac{dx}{r} N, \qquad (42)$$

where r is the probable error of a single observation.

Formula (42) affords a rough comparison of theory and experience without the use of tables. For instance, let the target-shots described in Art. 18 be again considered, and regard those in the middle division as having the error 0, those in the next division above as having the error $+1$, and so on. Then the errors, without regard to sign, are as in the first column below, their squares in the second, their weights or the number of shots in the third, and the weighted squares in the fourth.

x.	x^2.	p.	px^2.	p'.
0	0	212	0	261
1	1	394	394	382
2	4	282	1,128	232
3	9	89	801	93
4	16	20	320	26
5	25	3	75	6
			$\Sigma px^2 = 2,718$	1,000

Now, the probable error of a single observation is

$$r = 0.6745\sqrt{\frac{2718}{1000}} = 1.1,$$

and, by formula (42), the number of errors having the value 0 is

$$N_0 = \frac{0.2691 \times 1 \times 1000}{1.1} = 245,$$

which is a satisfactory agreement with the actual number 212. In the last column of the above table are given the theoretical numbers of errors as computed from Table II.

The Rejection of Doubtful Observations.

130. The theoretical distribution of errors, according to the fundamental formula (1), is shown by the values of the probability integral given in Table II; and from these it is seen, as in Art. 128, that the number of errors greater than $4r$ or $5r$ is very small. It becomes, then, a question, whether the probability of an error might not be so small that it would be justifiable to reject entirely the corresponding observation. For instance, if one thousand direct observations be taken, the probability that there will be one error greater than $5r$ is $\frac{1}{1000}$; if, then, in taking a series of, say, fifty observations, one error should exceed $5r$, the probability of its occurrence would be very much smaller than $\frac{1}{1000}$, and the observer would be tempted to reject that observation. But undoubtedly it would be a dangerous thing to allow an observer to decide upon his own limit of rejection. It has accordingly been proposed to attempt to establish a criterion by which the limit may be legitimately established from the principles of the probability of error. The criterion proposed by Chauvenet is the simplest of those deduced for this purpose, and is the following:

Let n be the number of direct observations, and also the number of errors. Let r denote the probable error of a single observation as found from the n residuals by formula (20). Let x be the limiting error, and let $\frac{x}{r}$ be called t. Let P be the value of the integral in Table II corresponding to t. Then

$$(43) \qquad P = \frac{2n-1}{2n}, \quad \text{and} \quad x = tr$$

is the criterion for the rejection of the largest residual.

To prove this, consider that the quantities in Table II need only be multiplied by the total number of errors to show the actual distribution; so that nP indicates the number of errors

less than x, and $n - nP$ indicates the number greater than x. Now, if

$$n - nP = \tfrac{1}{2},$$

there is but half an error greater than x, and any error greater than this x would be larger than allowed by the theoretical distribution. Hence the value of x corresponding to this value of P is the limiting value, which indicates whether the greatest residual in a series may be rejected or not.

131. In order to facilitate the use of this criterion, Table VII has been computed, giving the value of t directly for several values of n. For instance, if n is 5, the value of P is $\frac{10-1}{10}$, or 0.9; and from Table VII the corresponding value of t is 2.44.

The following particular example will illustrate the method of procedure. Let there be given thirteen observations of an angle, as in the first column below.

62° 12′ 51″.75	2.69	7.24
48.45	0.61	0.37
50.60	1.54	2.37
47.85	1.21	1.46
51.05	1.99	3.96
47.75	1.31	1.72
47.40	1.66	2.76
48.85	0.21	0.04
49.20	0.14	0.02
48.90	0.16	0.03
50.95	1.89	3.57
50.55	1.49	2.22
44.45	4.61	21.25
62° 12′ 49″.06		47.01

Let the mean of these be found, the residuals placed in the second column, and their squares in the third. The sum Σv^2 is 47.01; and hence, from (20), the probable error r of a single observation is $1''.32$. Table VII gives $t = 3.07$ when $n = 13$: hence, by the criterion, the limiting error is

$$x = 3.07 \times 1.32 = 4.05,$$

and accordingly the largest residual 4.61 should be rejected. To ascertain if the next largest residual, 2.99, should also be rejected, the mean of the twelve good observations should be found, and a new r computed from the twelve new residuals. But evidently the new sum Σv^2 will not differ greatly from the former sum minus the square of the rejected residual, or

$$\text{new } \Sigma v^2 = 47.01 - 21.25 = 25.76,$$

from which the new r is found to be about $1''.03$. Then the limiting error is

$$x = 3.02 \times 1.03 = 3''.11,$$

which shows that the residual 2.99 is not to be rejected.

132. Hagen's deduction of the law of probability of error, given in Chap. II, suggests another method of finding the limiting error of observation, and a new criterion for rejection. In Art. 26 the maximum error is expressed by $m\Delta x$, and the quantity $m\Delta x^2$ is replaced by $\frac{1}{2h^2}$. It is hence easy, by the help of (17), to find

$$(44) \qquad m\Delta x = 4.4 \frac{r^2}{dx},$$

where dx is the constant interval between successive values of the errors. For the observations discussed in Art. 129 this formula gives the limiting error $m\Delta x$ as 5.3, which seems entirely satisfactory. It is not possible to apply it, however, to angle measurements like those of the last article, on account of the impossibility of assigning a proper value to the interval dx.

The same difficulty prevents the practical use of formula (42), except in cases where this constant interval is definitely known.

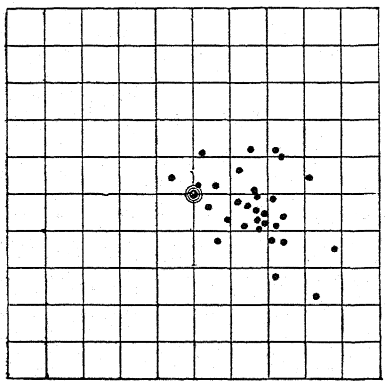
There is another criterion, due to Peirce, which may be applied to the case of indirect observations involving several unknown quantities, as well as to that of direct measurements; but its development cannot be given here. In general, it should be borne in mind that the rejection of measurements for the single reason of discordance with others is not usually justifiable unless that discordance is considerably more than indicated by the criterions. A mistake is to be rejected, and an observation giving a residual greater than $4r$ or $5r$ is to be regarded with suspicion, and be certainly rejected if the note-book shows any thing unfavorable in the circumstances under which it was taken. Usually, in practice, the number of large errors is greater than should be the case, according to theory; and this seems to indicate, either that the series is not sufficiently extended to give a reliable value of r, or that abnormal causes of error affect certain observations. If it were possible to increase the number of measurements, it would undoubtedly be found that the abnormal errors would be as often positive as negative, and that, for a very great number, there would be few that could be rejected by the criterion.

Constant Errors.

133. In all that has preceded, it has been supposed that the constant errors of observation have been eliminated from the numerical results before discussing them by the Method of Least Squares. If this is not done, and all the measurements of a set are affected by the same constant error, that error will also appear in the adjusted result. For instance, suppose thirty shots to be fired with the intention of hitting the centre of a target, and let their actual distribution be as shown in the figure. The most probable location of the centre, according to the records, is about two spaces to the right, and about half a

space below the true centre. Each shot, then, has been subject to these constant errors; the first due, perhaps, to the wind, and the second to gravity. If, now, these marks on the target represented observations for the purpose of locating the centre, the result obtained by their adjustment would be in error by the amounts just stated. Therefore, if all the observations of a series are affected by the same constant error, the Method of Least Squares can do nothing but adjust the accidental errors; and the probable errors of the adjusted results refer only to them, and give no indication whether constant causes of error affect the measurements or not.

Fig. 13.



134. The probability of the existence of a constant error in a case like that just illustrated is evidently large, and the numerical probability of its value lying between certain limits may be found by the help of Table II. The following is an example of such a discussion:

Suppose that an angle is laid out with very accurate instruments, and tested in many ways, so that its true value may be regarded as exactly 90°. Let twenty-five observations be taken upon it with a transit whose accuracy is to be tested, and let the mean of those measurements be 89° 59′ 57″ ± 0″.8. Then it is extremely probable that a constant error of about − 3″ exists in the instrument. To find the numerical expression of this probability, suppose that the true value of the angle was unknown, and ask the probability that the mean is within 2″ of the truth. Then, for $\frac{x}{r} = \frac{2}{0.8} = 2.5$, the value of the integral in

Table II is 0.908; so that it is a wager of 908 to 92, or of almost 10 to 1, that the mean is between the limits 89° 59′ 55″ and 89° 59′ 59″. Hence, since the angle is known to be 90°, it must be the same probability and the same wager that there is a constant error lying between the limits − 1″ and − 5″. So, also, if $x = 3''$, it may be shown that it is a wager of 39 to 1 that there is a constant error between 0″ and − 6″.

135. In case that several sources of constant error exist, the adjustment by the Method of Least Squares tends to eliminate them, and to give results nearer and nearer to the actual values, as the number of observations is increased. This will be rendered evident by considering again the illustration of the target. One marksman fires thirty balls, which are subject to a constant error, as in Fig. 13. Another marksman fires thirty more, which have a different constant error, owing to the peculiarities in his aim. A third marksman has a third constant error, in a still different direction. The shots of each marksman are distributed around their most probable centre in accordance with the law of probability of accidental errors. And undoubtedly these constant errors will be grouped around the true centre according to the same law; and, as the number of marksmen increases, the constant errors will thus tend to annul each other, and ultimately make the most probable centre coincide with the true one.

And so it must be in angle observations, when great precision is demanded. On one day certain constant errors, due to atmospheric influences, affect all results in a certain direction; while on a second day, under different influences, new constant errors act in another direction. If the measurements be continued over many days, the number and magnitude of positive constant errors will be likely to equal the negative ones; so that the adjustment by the Method of Least Squares will balance them, and give results near to the true values.

Here may be seen the reason why the number of large residuals is usually greater than the theory demands, and also a reason why a criterion for rejection cannot generally be safely applied to series of observations consisting of few measurements.

Social Statistics.

136. It is found that the law of probability of error applies to many phenomena of social and political science. If men be arranged in groups, according to their heights, there will be found few dwarfs and few giants; and the numbers in the different groups will closely agree with the theoretical distribution required by the curve of probability. The following table, which is taken from Gould's Statistics (New York, 1869),

Height. Inches.	Actual Number.	Proportional Number in 10,000.		
		Observed.	Calculated.	Calc. — Obs.
61	197	105	100	− 5
62	317	169	171	+ 2
63	692	369	368	− 1
64	1289	686	675	− 11
65	1961	1044	1051	+ 7
66	2613	1391	1399	+ 8
67	2974	1584	1584	0
68	3017	1607	1531	− 76
69	2287	1218	1260	+ 42
70	1599	852	884	+ 32
71	878	467	531	+ 64
72	520	277	267	− 10
73	262	139	118	− 21
74	174	92	61	− 31

gives a comparison of the theoretical and observed heights of 18,780 white soldiers, including men of all nativities and ages. In the second column are recorded the actual number measured of each height, and, in the third, the proportional number in 10,000. The mean height as found by formula (9) is 67.24 inches, from this the residuals are formed; and the probable error of a single determination, by formula (23), is 1.676 inches. The theoretical numbers between the several limits are next derived by the help of Table II, and recorded in the fourth column, while the differences between the calculated and observed numbers are given in the last.

137. Numerous comparisons of this kind, made by Quetelet and others, have clearly established that stature and the other proportions of the body are governed by the law of probability of error. Nature, in fact, aims to produce certain mean proportions; and the various groups into which mankind may be classified deviate from the mean according to the law of the probability curve. And the same is true of intellect. By the discussion of social statistics, then, it is possible to discover the mean type of humanity, not merely in physical proportion, but in intellect, capacity, judgment, and desires. "The average man," says Quetelet, "is for a nation what the centre of gravity is for a body: to the consideration of this are referred all the phenomena of equilibrium."

In fact, the distribution of social phenomena seems strictly analogous to that of the rifle-shots discussed in Art. 135. Each shot may represent a person, or some property of a person, to be investigated. For all the shots there is a mean, showing the most probable result; and also, for each group, there is a secondary mean, depending on the particular race or nation to which the person belongs. There is a type for soldiers, and another for sailors; one for Americans, and another for Europeans; one for men, and another for women; one for the period

of youth, and another for that of maturity. The individuals of each type are clustered around its mean, according to the law of probability; and the several types are clustered around a general mean, according to the same law. This is true for all statistical data in which equal positive and negative deviations from the mean are equally probable; in other cases an unsymmetric distribution may occur.

138. *Problems.*

1. An angle is measured by an instrument graduated to quarter-minutes, the probable error of a single reading being 12 seconds. How many observations are necessary, that it may be a wager of 5 to 1 that the mean is within one second of the truth?

2. A line is measured 500 times. If the probable error of each observation is 0.6 centimeters, how many errors will be less than 1 centimeter, and greater than 0.4 centimeters?

3. The capacity of a certain large vessel is unknown: 1,600 persons guess at the number of gallons of water which it will hold, and the average of their guesses is 289 gallons. The vessel is then measured by a committee, and found to hold 297 gallons. If the probable error of a single guess be 50 gallons, and it be impossible that there can be any constant source of error in guessing, what is the probability that the committee have an error in their measurement of between 3 and 13 gallons?

4. Determine from the data in Art. 136 the number of men per million who are more than seven feet tall.

5. Two observations differ by the amount a. A third observation differs from the mean of the first two by the amount u. Find, by Chauvenet's criterion, the value of u necessary to reject the third observation.

CHAPTER X.

SOLUTION OF NORMAL EQUATIONS.

139. In the preceding pages the student has been left to solve normal equations by any common algebraic process. It is usual in computing offices, however, to require them to be formed and solved by a definite method for the sake of uniformity in making comparisons. This is, indeed, absolutely necessary when the number of unknown quantities is greater than three or four, or when the co-efficients are large, in order that checks upon the numerical work may be constantly had and the accuracy of the results be ensured. The methods in most common use will now be explained.

Three Normal Equations.

140. The method of elimination, due to Gauss, which is described below, is probably the best for this case except when the co-efficients are small numbers. In that event the determinant formulas for solution may be advantageously employed. These will be here written for the general case of three linear equations,

$$A_1x + B_1y + C_1z = D_1,$$
$$A_2x + B_2y + C_2z = D_2,$$
$$A_3x + B_3y + C_3z = D_3,$$

the solution of which gives the formulas,

$$x = \begin{vmatrix} D_1 & B_1 & C_1 \\ D_2 & B_2 & C_2 \\ D_3 & B_3 & C_3 \end{vmatrix} \div \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix},$$

$$y = \begin{vmatrix} A_1 & D_1 & C_1 \\ A_2 & D_2 & C_2 \\ A_3 & D_3 & C_3 \end{vmatrix} \div \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix},$$

$$z = \begin{vmatrix} A_1 & B_1 & D_1 \\ A_2 & B_2 & D_2 \\ A_3 & B_3 & D_3 \end{vmatrix} \div \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}.$$

These are readily kept in mind by noticing that the denominator is the same for each, and that in the numerator the absolute terms D replace the co-efficients of the unknown quantity to be found. If $C_1 = C_2 = C_3 = 0$, and $A_3 = B_3 = D_3 = 0$, this solution reduces to that given in Art. 55.

141. As an illustration of this method let the three normal equations be

$$3x - y + 2z = 5,$$
$$-x + 4y + z = 6,$$
$$2x + y + 5z = 3.$$

Then the determinant denominator, being developed, gives

$$\begin{vmatrix} 3 & -1 & 2 \\ -1 & 4 & 1 \\ 2 & 1 & 5 \end{vmatrix} = 3\begin{vmatrix} 4 & 1 \\ 1 & 5 \end{vmatrix} + 1\begin{vmatrix} -1 & 2 \\ 1 & 5 \end{vmatrix} + 2\begin{vmatrix} -1 & 2 \\ 4 & 1 \end{vmatrix} = 32.$$

Similarly the values of the three determinant numerators are found to be 110, 86, and -42. Hence

$$x = +\tfrac{55}{16}, \qquad y = +\tfrac{43}{16}, \qquad z = -\tfrac{21}{16},$$

which exactly satisfy the three given normal equations.

Checks upon the results of the solution may be also obtained by writing the normal equations in another order, making for instance the third the first one, and thus obtaining different numerical determinants for development.

Formation of Normal Equations.

142. Let the n observation equations between three unknown quantities be of equal weight, and let the observed quantities $M_1, M_2, \ldots M_n$ be transposed to the first term, giving

$$a_1x + b_1y + c_1z + m_1 = 0,$$
$$a_2x + b_2y + c_2z + m_2 = 0,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$a_nx + b_ny + c_nz + m_n = 0,$$

and let there be formed the sums

$$a_1^2 + a_2^2 + \ldots + a_n^2 = [aa],$$
$$a_1b_1 + a_2b_2 + \ldots + a_nb_n = [ab].$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

Then the three normal equations are

$$[aa]x + [ab]y + [ac]z + [am] = 0,$$
$$[ba]x + [bb]y + [bc]z + [bm] = 0,$$
$$[ca]x + [cb]y + [cc]z + [cm] = 0.$$

Thus the formation of the normal equations consists in computing the co-efficients $[aa]$, $[ab]$, etc. This may be done by common arithmetic, by the help of Crelle's multiplication table, a logarithmic table, a table of squares, or a calculating machine. The following method of arranging and checking the work is frequently employed.

Write the co-efficients and absolute terms of the observation

equations in tabular form and add a column containing the algebraic sums of these for each equation. Thus for three

	x	y	z		
	a	b	c	m	s
1					
2					
.					
.					
.					
n					

unknown quantities the table has the above form, the last column containing, for each horizontal row, the algebraic sum $a + b + c + m = s$.

A second table, which need not be here shown, contains fifteen columns, headed aa, ab, . . . ss, and the summation of the products in these columns gives the fifteen co-efficients and absolute quantities which are arranged in a third table as below. It is to be noted that $[ba]$, $[ca]$, $[cb]$ are the same as $[ab]$, $[ac]$, $[bc]$, and hence need not be computed.

	x	y	z			Check.
	$a]$	$b]$	$[c$	$m]$	$s]$	
$[a$						
$[b$						
$[c$						
$[m$						
$[s$						

Here the sum $[bb]$ is placed at the right of $[b$ and under $b]$, the

sum $[cs]$ at the right of $[c$ and under $s]$, and so on. The last column is used to record the results of the five checks, namely,

$$[aa] + [ab] + [ac] + [am] = [as],$$
$$[ba] + [bb] + [bc] + [bm] = [bs],$$
$$[ca] + [cb] + [cc] + [cm] = [cs],$$
$$[ma] + [mb] + [mc] + [mm] = [ms],$$
$$[sa] + [sb] + [sc] + [sm] = [ss].$$

If these checks are all fulfilled, the normal equations may be regarded as correctly formed. In filling out the table the coefficients $[ba]$, $[mc]$, etc., need not be written, since they are the same as $[ab]$, $[cm]$, etc.

143. As a simple example let five observations upon three quantities give the five observation equations

$$\begin{array}{rrrrl} -x & & +z & -2 & = 0, \\ -x & +y & & -9 & = 0, \\ & +y & & -18 & = 0. \\ & +y & -z & -7 & = 0, \\ & & +z & -10 & = 0. \end{array}$$

The arrangement of the first table is then as follows:

No.	x a	y b	z c	m	s
1	-1	0	$+1$	-2	-2
2	-1	$+1$	0	-9	-9
3	0	$+1$	0	-18	-17
4	0	$+1$	-1	-7	-7
5	0	0	$+1$	-10	-9

The products aa, ab, etc., are next computed, and the sums $[aa]$, $[ab]$, etc., are found. The table of co-efficients and absolute quantities then is

	x $a]$	y $b]$	z $c]$	$m]$	$s]$	Check.
$[a$	$+2$	-1	-1	$+11$	$+11$	$+11$
$[b$		$+3$	-1	-34	-33	-33
$[c$			$+3$	-5	-4	-4
$[m$				$+558$	$+530$	$+530$
$[s$					$+504$	$+504$

and the checks being all fulfilled the computations are satisfactory. Thus the normal equations are

$$
\begin{aligned}
+2x - y - z + 11 &= 0, \\
-x + 3y - z - 34 &= 0, \\
-x - y + 3z - 5 &= 0,
\end{aligned}
$$

and it will be shown in Art. 147 how these may be solved so as to continue the above system of checks throughout the entire numerical work.

The advantage of the above system is more apparent in cases where the co-efficients and absolute terms consist of several digits and where the decimals must be rounded off. In such cases the number of decimals to be retained in the work should be at least sufficient to cause the checks to be fulfilled with an error not greater than one unit in the last place. The additional labor required for these checks is fully repaid by the assurance of correctness in the numerical work.

Gauss's Method of Solution.

144. The method of solution due to Gauss, by which is preserved throughout the work the symmetry that exists in the coefficients of the normal equations, is extensively used by computors. To illustrate it, three normal equations of equal weight will be sufficient.

From the n observation equations are derived, by the method of Art. 142, the three normal equations

$$[aa]x + [ab]y + [ac]z + [am] = 0,$$
$$[ba]x + [bb]y + [bc]z + [bm] = 0,$$
$$[ca]x + [cb]y + [cc]z + [cm] = 0.$$

From the first equation take the value of x and substitute it in the second and third, giving

$$\left([bb] - \frac{[ba][ab]}{[aa]}\right)y + \left([bc] - \frac{[ba][ac]}{[aa]}\right)z + \left([bm] - \frac{[ba][am]}{[aa]}\right) = 0,$$
$$\left([bc] - \frac{[ba][ac]}{[aa]}\right)y + \left([cc] - \frac{[ca][ac]}{[aa]}\right)z + \left([cm] - \frac{[ca][am]}{[aa]}\right) = 0.$$

For the sake of abbreviation the quantities within the parentheses may be denoted by $[bb.1]$, $[bc.1]$, $[bm.1]$ for the first equation, and by $[cb.1]$, $[cc.1]$, $[cm.1]$ for the second equation. Then these two equations may be written

$$[bb.1]y + [bc.1]z + [bm.1] = 0,$$
$$[cb.1]y + [cc.1]z + [cm.1] = 0,$$

which are similar in form to the second and third normal equations, except that the terms containing x have disappeared and each co-efficient is marked with a 1. These quantities, $[bb.1]$, $[bc.1]$, may be called "auxiliaries," and the law of their formation is evident.

From the first of these equations take the value of y and substitute it in the second, giving

$$\left([cc\,.\,1] - \frac{[cb\,.\,1][bc\,.\,1]}{[bb\,.\,1]}\right)z + \left([cm\,.\,1] - \frac{[cb\,.\,1][bm\,.\,1]}{[bb\,.\,1]}\right) = 0,$$

which may be abbreviated into

$$[cc\,.\,2]z + [cm\,.\,2] = 0,$$

where $[cc\,.\,2]$ and $[cm\,.\,2]$ may be called "second auxiliaries."

The value of the quantity z now is

$$z = -\frac{[cm\,.\,2]}{[cc\,.\,2]},$$

while the values of y and x are

$$y = -\frac{[bm\,.\,1]}{[bb\,.\,1]} - \frac{[bc\,.\,1]}{[bb\,.\,1]}z,$$

$$x = -\frac{[am]}{[aa]} - \frac{[ac]}{[aa]}z - \frac{[ab]}{[aa]}y;$$

and the correctness of these results may be tested by inserting the computed values of x, y, z in the second and third normal equations. Or the order of computation may be reversed and the value of x be first obtained, z being first eliminated and then y; this will be necessary only in critical cases.

145. When the normal equations have been formed by the method of Art. 142, the checks there explained should be continued by the computation of the auxiliaries $[mm\,.\,1]$, $[bs\,.\,1]$, etc.; thus,

$$[bs\,.\,1] = [bs] - \frac{[ba][as]}{[aa]}.$$

And a second table should be formed for the two equations containing y and z, by which four numerical checks are obtained.

In the next step also the auxiliaries $[mm.2]$, $[cs.2]$, etc., are found; for example,

$$[cs.2] = [cs.1] - \frac{[cb.1][bs.1]}{[bb.1]},$$

and then the third table affords three numerical checks.

146. A valuable final check is obtained by computing the third set of auxiliaries; thus,

$$[mm.3] = [mm.2] - \frac{[mc.2][cm.2]}{[cc.2]},$$

$$[ms.3] = [ms.2] - \frac{[mc.2][cs.2]}{[cc.2]},$$

$$[ss.3] = [ss.2] - \frac{[sc.2][cs.2]}{[cc.2]},$$

and these three values are equal. Each is also equal to the quantity Σv^2, or to the sum of the squares of the residuals obtained by substituting in the observation equations the values of x, y, and z, found from the normal equations.

To prove this let an observation equation be

$$ax + by + cz + m = 0.$$

Then the most probable values, x, y, z, will not reduce it to zero, but leave a small residual v. Hence, strictly,

$$ax + by + cz + m = v.$$

By squaring each of the values of v, and adding the results, the value of Σv^2 is found; and if from this each normal equation, first multiplied by its unknown quantity, be subtracted, it reduces to

$$[am]x + [bm]y + [cm]z + [mm] = \Sigma v^2.$$

If this be regarded as a fourth normal equation, it becomes, after the elimination of x,

$$[bm.1]y + [cm.1]z + [mm.1] = \Sigma v^2,$$

and after eliminating y it is

$$[cm.2]z + [mm.2] = \Sigma v^2;$$

and finally, after the elimination of z,

$$[mm.3] = \Sigma v^2.$$

Hence the auxiliary $[mm.3]$ is equal to the sum of the squares of the residuals; and that $[ms.3]$ and $[ss.3]$ have the same value is shown by the method of their formation.

147. As a simple numerical example let the following observation equations, all of weight unity, be taken:

$$\begin{aligned}
-x \qquad\quad + z - 2 &= 0,\\
-x + y \qquad\quad - 9 &= 0,\\
+ y \qquad\quad - 18 &= 0,\\
+ y - z - 7 &= 0,\\
+ z - 10 &= 0.
\end{aligned}$$

The normal equations for this case have already been formed in Art. 143, and the values of its co-efficients and check numbers will be taken from the table there given.

The computation of the auxiliaries for the two equations containing y and z is now made, thus:

$$[bb.1] = [bb] - \frac{[ba][ab]}{[aa]} = + \quad 3 - \frac{1 \times 1}{2} = + 2.5,$$

$$[bc.1] = [bc] - \frac{[ba][ac]}{[aa]} = - \quad 1 - \frac{1 \times 1}{2} = - 1.5,$$

$$[bm.1] = [bm] - \frac{[ba][am]}{[aa]} = -\ 34 + \frac{1 \times 11}{2} = -28.5,$$

$$[bs.1] = [bs] - \frac{[ba][as]}{[aa]} = -\ 33 + \frac{1 \times 11}{2} = -27.5,$$

$$[cc.1] = [cc] - \frac{[ca][ac]}{[aa]} = +\ 3 - \frac{1 \times 1}{2} = +2.5,$$

$$[cm.1] = [cm] - \frac{[ca][am]}{[aa]} = -\ 5 + \frac{1 \times 11}{2} = +0.5,$$

$$[cs.1] = [cs] - \frac{[ca][as]}{[aa]} = -\ 4 + \frac{1 \times 11}{2} = +1.5,$$

$$[mm.1] = [mm] - \frac{[ma][am]}{[aa]} = +558 - \frac{11 \times 11}{2} = +497.5,$$

$$[ms.1] = [ms] - \frac{[ma][as]}{[aa]} = +530 - \frac{11 \times 11}{2} = +469.5,$$

$$[ss.1] = [ss] - \frac{[sa][as]}{[aa]} = +504 - \frac{11 \times 11}{2} = +443.5,$$

and the corresponding tabulation is as follows, the four checks being exactly fulfilled:

	y $b.1]$	z $c.1]$	$m.1]$	$s.1]$	Check.
$[b$	$+2.5$	-1.5	-28.5	-27.5	-27.5
$[c$		$+2.5$	$+0.5$	$+1.5$	$+1.5$
$[m$			$+497.5$	$+469.5$	$+469.5$
$[s$				$+443.5$	$+443.5$

The coefficient $[cc.2]$ and the auxiliaries for the final equation in z are next found; thus,

$$[cc.2] = [cc.1] - \frac{[cb.1][bc.1]}{[bb.1]} = +1.6,$$

$$[cm.2] = [cm.1] - \frac{[cb.1][bm.1]}{[bb.1]} = -16.6,$$

$$[cs.2] = [cs.1] - \frac{[cb.1][bm.1]}{[bb.1]} = -15.0,$$

$$[mm.2] = [mm.1] - \frac{[mb.1][bm.1]}{[bb.1]} = +172.6,$$

$$[ms.2] = [ms.1] - \frac{[mb.1][bs.1]}{[bb.1]} = +156.0,$$

$$[ss.2] = [ss.1] - \frac{[sb.1][bs.1]}{[bb.1]} = +141.0,$$

and the corresponding table with its checks is:

	z $c.2]$	$m.2]$	$s.2]$	Check.
$[c$	$+1.6$	-16.6	-15.0	-15.0
$[m$		$+172.6$	$+156.0$	$+156.0$
$[s$			$+141.0$	$+141.0$

The value of the unknown quantity z now is

$$z = -\frac{-16.6}{1.6} = +10.375,$$

and from the two equations containing y and z,

$$y = \frac{28.5}{2.5} + \frac{1.5}{2.5}z = +17.625,$$

and finally, from the first normal equation,

$$x = -\tfrac{11}{2} + \tfrac{1}{2}z + \tfrac{1}{2}y = +8.500.$$

These values also exactly satisfy the second and third normal equations.

Lastly, the final check of Art. 146 is applied by computing the third set of auxiliaries and the sum of the squares of the

residuals. There are found $[mm.3] = 0.375$, $[ms.3] = 0.375$, and $[ss.3] = 0.375$. Also, by substituting the values of x, y, z, in the observation equations,

$$v_1 = -0.125,\quad v_2 = +0.125,\quad v_3 = -0.375,\quad v_4 = +0.250,\quad v_5 = +0.375,$$

the sum of whose squares is $\Sigma v^2 = 0.375$. Hence the correctness of all the numerical work is assured.

When the coefficients of the normal equations contain decimals these are to be rounded off as the work progresses, so that the checks may be sufficiently satisfied.

Weighted Observations.

148. The method of Gauss is also directly applicable to normal equations derived from independent weighted observation equations. The process will be illustrated for three unknown quantities. Let the observation equations be

$$\begin{array}{lll}
+x & = 0, & p_1 = 85, \\
\quad +y & = 0, & p_2 = 108, \\
\qquad +z & = 0, & p_3 = 49, \\
+x-y \quad +0.92 & = 0, & p_4 = 165, \\
\quad -y+z+1.35 & = 0, & p_5 = 78, \\
-x \quad +z+1.00 & = 0, & p_6 = 60.
\end{array}$$

The first table is then as follows :

No.	p	x a	y b	z c	m	s
1	85	+1	0	0	0	+1
2	108	0	+1	0	0	+1
3	49	0	0	+1	0	+1
4	165	+1	−1	0	+0.92	+0.92
5	78	0	−1	+1	+1.35	+1.35
6	60	−1	−0	+1	+1	+1

Next the co-efficients [*paa*], [*pab*], etc., are computed and the table of normal equations is formed, the co-efficients below the diagonal line being omitted, since [*pba*] is the same as [*pab*], and so on.

	x *a*]	y *b*]	z *c*]	*m*]	*s*]	Check.
[*pa*	+ 310	− 165	− 60	+ 91.8	+176.8	+176.8
[*pb*		+ 351	− 78	−257.1	−149.1	−149.1
[*pc*			+ 187	+165.3	+214.3	+214.3
[*pm*				+341.8	+341.8	+341.9
[*ps*					+583.8	+583.8

This shows by its checks that the computations are correct, the discrepancy between 341.8 and 341.9 being due to the rounding off of decimals. Thus,

$$\begin{aligned} 310x - 165y - 60z + 91.8 &= 0, \\ -165x + 351y - 78z - 257.1 &= 0, \\ -60x - 78y + 187z + 165.3 &= 0, \end{aligned}$$

are the normal equations for determining the most probable values of x, y, and z.

149. The auxiliaries [*pbb*.1], [*pbc*.1], etc., are computed by exactly the same rules as before, and the table for the two

	y *b*.1]	z *c*.1]	*m*.1]	*s*.1]	Check.
[*pb*	+ 263.2	− 109.9	− 208.2	− 55.0	− 54.9
[*pc*		+ 175.4	+ 183.1	+ 248.5	+ 248.6
[*pm*			+ 314.5	+ 289.4	+ 289.4
[*ps*				+ 483.0	+ 482.9

reduced normal equations containing y and z is formed, the four checks being fulfilled within one unit in the last figure.

The second auxiliaries $[pcc.2]$, $[pcm.2]$, etc., are computed exactly as before and the table for the final equation in z is

	z $c.2]$	$m.2]$	$s.2]$	Check.
$[pc$	$+129.5$	$+96.1$	$+225.5$	$+225.6$
$[pm$		$+149.8$	$+245.9$	$+245.9$
$[ps$			$+471.5$	$+471.4$

formed and its checks found to be satisfactory. The value of z now is

$$z = -\frac{96.1}{129.5} = -0.7421,$$

which is carried to four decimals in order that y and x may be found correct to two decimals.

From the first equations in the two tables preceding the last, the values of y and x are now obtained, thus,

$$y = +\frac{208.2}{263.2} + \frac{109.9}{263.2}z = +0.480,$$

$$x = -\frac{91.8}{310} + \frac{60}{310}z + \frac{165}{310}y = -0.18,$$

and hence the final results to two decimals are

$$x = -0.18, \quad y = +0.48, \quad z = -0.74,$$

which are the most probable values of the unknown quantities.

150. Inserting these values of x, y, z in the six observation equations, the residuals are found to be

$$v_1 = -0.18, \quad v_2 = +0.48, \quad v_3 = -0.74, \quad v_4 = +0.26,$$
$$v_5 = +0.13, \quad v_6 = +0.44.$$

Squaring these and multiplying each square by its corresponding weight there results

$$\Sigma pv^2 = 78.57.$$

The computation of the third auxiliaries gives

$$[pmm.3] = 78.5, \quad [pms.3] = 78.6, \quad [pss.3] = 78.8,$$

an agreement which is as close as is necessary for this case.

Logarithmic Computations.

151. The use of logarithms is often advantageous in forming the products required in the solution of normal equations. A systematic scheme for such solutions will now be presented in which the four-place logarithmic table given at the end of this volume will be employed. In general a five- or seven-place table will be found easier to use when the co-efficients contain more than four significant figures.

The scheme to be used will be as follows for three normal equations, the space for checks being in a horizontal row at the bottom and these checks referring to the auxiliaries instead of to the normal equations themselves, which are supposed to have been first formed and checked by the method of Art. 144. The form is first to be filled out by writing the numbers $[aa]$, $[ab]$, . . . $[ms]$ in the places indicated. The logarithms of $[aa]$, $[ab]$, . . . $[as]$ are next taken out and recorded. Then writing $\log [aa]$ on a strip of paper, it is subtracted in turn from $\log [ab]$, $\log [ac]$, $\log [am]$, $\log [as]$, and the differences are written, thus filling out the top row of squares.

Log $[ab]$ is now written on a slip of paper and added to the logarithms at the foot of the first row, thus giving the logarithms for the second row. Those in the third and fourth rows are similarly found by adding $\log [ac]$ and $\log [am]$ to the same ones as before. The numbers corresponding to

x	y	z	m	s
$[aa]$ $\log [aa]$	$[ab]$ $\log [ab]$ $\log \frac{[ab]}{[aa]}$	$[ac]$ $\log [ac]$ $\log \frac{[ac]}{[aa]}$	$[am]$ $\log [am]$ $\log \frac{[am]}{[aa]}$	$[as]$ $\log [as]$ $\log \frac{[as]}{[aa]}$
	$[bb]$ $\log \frac{[ab]}{[aa]}[ab]$ number $[bb.1]$	$[bc]$ $\log \frac{[ac]}{[aa]}[ab]$ number $[bc.1]$	$[bm]$ $\log \frac{[am]}{[aa]}[ab]$ number $[bm.1]$	$[bs]$ $\log \frac{[as]}{[aa]}[ab]$ number $[bs.1]$
		$[cc]$ $\log \frac{[ac]}{[aa]}[ac]$ number $[cc.1]$	$[cm]$ $\log \frac{[am]}{[aa]}[ac]$ number $[cm.1]$	$[cs]$ $\log \frac{[as]}{[aa]}[ac]$ number $[cs.1]$
			$[mm]$ $\log \frac{[am]}{[aa]}[am]$ number $[mm.1]$	$[ms]$ $\log \frac{[as]}{[aa]}[am]$ number $[ms.1]$
Checks.	$[bs.1]$	$[cs.1]$	$[ms.1]$	$[ss.1]$ $[ss.1]$

these logarithms are then taken from the table, and each number being subtracted from that at the top of the square, the co-efficients $[bb\ 1]$, $[bc.1]$, . . . $[ms.1]$ result. Lastly the check $[bs.1]$ at the foot of the second column is found by

adding together $[bb.1]$, $[bc.1]$, and $[bm.1]$; and in a similar manner $[cs.1]$ and $[ms.1]$ are found. Here $[ss.1]$ may be determined in two ways, by the addition of the horizontal row and also by the column above it.

A second similar tabulation is also made for the next operation, the auxiliaries $[bb.1]$, $bc.1]$, ... $[ms.1]$ being transferred from the first table to the top of the squares in the second one. The process will be now exemplified by a numerical example.

152. Let there be given three normal equations which have arisen from a case of conditioned observations, namely,

$$+17.73x - 4.80y - 8.13z + 4.60 = 0,$$
$$-4.80x + 17.60y - 2.40z + 34.89 = 0,$$
$$-8.13x - 2.40y + 13.93z - 7.75 = 0.$$

Here the check sums $[as]$, $[bs]$, $[cs]$, $[ms]$ are to be formed from the given co efficients; for example,

$$[cs] = -8.13 - 2.40 + 13.93 - 7.75 = -4.35,$$

but $[mm]$, $[ms]$, and $[ss]$ cannot be obtained. For the purpose of carrying through the full system of checks, one of these, say $[mm]$, may be assumed, and the others be computed; assuming $[mm] = 0$, the value of $[ms]$ is $+31.74$. The co-efficients and check numbers are then arranged in the upper right-hand corners of the squares in the following table. The four-place logarithms of those in the upper row are taken out, the letter n being affixed to the logarithm of a negative number. The subtractions and additions of these logarithms as required by the scheme of the last article are then made, and the corresponding numbers taken from the logarithmic table. These subtracted from those in the upper corners give the auxiliaries $[bb.1]$, $[bc.1]$, etc., which are written in the lower

right-hand corners. The checks of these are then made, and found to be verified to one unit of the last decimal.

x	y	z	m	s
+ 17.73 1.2487	− 4.80 0.6812n $\bar{1}$.4325n	− 8.13 0.9101n $\bar{1}$.6614n	+ 4.60 0.6628 $\bar{1}$.4141	+ 9.40 0.9731 $\bar{1}$.7244
	+ 17.60 0.1137 + 1.30 + 16.30	− 2.40 0.3426 + 2.20 − 4.60	+ 34.89 0.0953n − 1.25 + 36.14	+ 45.29 0.4056n − 2.54 + 47.83
		+ 13.93 0.5715 + 3.73 + 10.20	− 7.75 0.3242n − 2.11 − 5.64	− 4.35 0.6345n − 4.31 − 0.04
			+ 0.00 0.0769 + 1.19 − 1.19	+ 31.74 0.3872 + 2.44 + 29.30
Checks	+ 47.84	− 0.05	+ 29 31	+ 77.09 + 77.10

The next operation is to write the values of the auxiliaries [bb. 1], [bc. 1], ... [ms. 1] in a second table of squares, and by a similar process obtain the second set [cc. 2], ... [ms. 2].

The scheme shown in the twelve upper left-hand squares of the table in Art. 151 will apply to this case if a, b, c, m be changed to b, c, m, s, and 1 added in all brackets except the

y	z	m	s
+ 16.30	− 4.60	+ 36.14	+ 47.83
1.2122	0.6628n	1.5580	1.6797
	$\bar{1}$.4506n	0.3458	0.4675
	+ 10.20	− 5.64	− 0.04
	0.1134	1.0086n	1.1303n
	+ 1.30	− 10.20	− 13.50
	+ 8.90	+ 4.56	+ 13.46
		− 1.19	+ 29.30
log 4.56	= 0.6590	1.9038	2.0255
log 8.90	= 0.9494	+ 80.13	+ 106.06
log z	= $\bar{1}$.7096n	− 81.32	− 76.76
			− 63.30
Check.	+ 13.46	− 76.76	− 63.30

lowest in each square where the 1 is changed to 2. The operations are strictly analogous to those of the preceding table.

A table for the computation of the third set of auxiliaries need not be formed, these being of no use, as the sum $[mm]$ was assumed at the start. The value of z now is

$$z = -\frac{4.56}{8.90} \quad \text{or} \quad z = -\log^{-1} \bar{1}.7096 = -0.512.$$

From the logarithms in the upper squares of the last table,

$$y = -\log^{-1}(0.3458) - \log^{-1}(\bar{1}.4506n + \bar{1}.7096n) = -2.362,$$

and similarly from the logarithms in the upper squares of the first table, according to the last formula of Art. 144,

$$x = -\log^{-1}(\bar{1}.4141) - \log^{-1}(\bar{1}.6614n + \bar{1}.7096n)$$
$$- \log^{-1}(\bar{1}.4325n + 0.3731n) = -1.134,$$

which are the values that closely satisfy the given normal equations.

After becoming acquainted with this method by solving several sets of normal equations the student will find it, except when the coefficients are small integers, to be generally more expeditious than methods which do not employ logarithms.

Probable Errors of Adjusted Values.

153. When the sum of the weighted squares of the residuals, Σpv^2, has been computed, the probable error of an independent observation of weight unity is given by (32), namely,

$$r = 0.6745\sqrt{\frac{\Sigma pv^2}{n-q}},$$

in which n is the number of independent observations and q the number of unknown quantities. If p_x, p_y, p_z be the weights of the adjusted values of x, y, z, the probable errors of these adjusted values then are

$$r_x = \frac{r}{\sqrt{p_x}}, \quad r_y = \frac{r}{\sqrt{p_y}}, \quad r_z = \frac{r}{\sqrt{p_z}},$$

and thus these are known as soon as the weights have been determined.

154. To find these weights the methods of Arts. 74, 75 may be conveniently employed for three unknown quantities. Using the solution in Art. 141, replacing A_{1}, B_{1}, C_{1}, etc., by $[aa]$, $[ab]$, $[ac]$, etc., and designating by D the determinant denominator common to the three values, there are found,

$$p_x = \frac{D}{[bb][cc] - [bc]^2}, \quad p_y = \frac{D}{[aa][cc] - [ac]^2}, \quad p_z = \frac{D}{[aa][bb] - [ab]^2},$$

which are the weights of the adjusted values of x, y, z.

Referring again to Art. 74, and to the method of Gauss given in Art. 144, it is seen that the value of z is

$$z = -\frac{[cm\,.\,2]}{[cc\,.\,2]}.$$

The negative sign here results from the fact that the absolute terms $[am]$, $[bm]$, etc., are taken positive in the first members of the normal equations, and the numerator vanishes when those terms are all zero. The quantity $[cc\,.\,2]$ is thus the reciprocal of the co-efficient of the absolute terms which belonged to the normal equation for z and is hence the weight of z, or $p_z = [cc\,.\,2]$.

By equating this value of p_z to that found above, D may be eliminated from the three expressions, giving

$$p_z = [cc\,.\,2], \quad p_y = \frac{[cc\,.\,2][bb\,.\,1]}{[cc\,.\,1]}, \quad p_x = \frac{[cc\,.\,2][bb\,.\,1][aa]}{[bb][cc] - [bc]^2},$$

which are values of the weights expressed in terms of the coefficients and auxiliaries used in finding the value of x, y, z.

155. For example, consider the six observation equations of Art. 148, and let it be required to find the probable errors of the adjusted values of x, y, z. The normal equations are

solved in Art. 149, giving $x = -0.18$, $y = +0.48$, $z = -0.74$, and the value of Σpv^2 is found to be 78.6; thus,

$$r = 0.6745\sqrt{\frac{78.6}{6-3}} = 3.45$$

is the probable error of an observation of weight unity. The weights of the adjusted values of x, y, z are

$$p_z = 129.5, \quad p_y = \frac{129.5 \times 263.2}{175.4} = 194.4,$$

$$p_x = \frac{129.5 \times 263.2 \times 310}{351 \times 187 - 78^2} = 177.5,$$

and the probable errors of the values of x, y, z are

$$r_x = \frac{3.45}{\sqrt{177.5}} = 0.26, \quad r_y = \frac{3.45}{\sqrt{194.4}} = 0.25, \quad r_z = \frac{3.45}{\sqrt{129.5}} = 0.30.$$

Accordingly the adjusted values may be written

$$x = -0.18 \pm 0.26, \quad y = +0.48 \pm 0.25, \quad z = -0.74 \pm 0.30,$$

which shows the degree of mental confidence that the adjusted values may claim.

156. When the number of unknowns is large the expressions for the weights of the adjusted values become quite complex, and in order to find their values it may be sometimes advisable to deduce x, y, z, w, etc., by two or more different orders of elimination. The following are formulas for the weights for the case of four unknown quantities, where w is first determined and x last:

$$p_w = [dd.3], \qquad p_z = \frac{[dd.3][cc.2]}{[dd.2]},$$

$$p_y = \frac{[dd.3][cc.2][bb.1]}{[dd.2]_b[cc.1]}, \quad p_x = \frac{[dd.3][cc.2][bb.1][aa]}{[dd.2]_a[cc.1]_a[bb]},$$

in which the subscript quantities have the following values,

$$[cc.1]_a = [cc] \quad - \frac{[bc]^2}{[bb]},$$

$$[dd.2]_b = [dd.1] - \frac{[cd.1]^2}{[cc.1]},$$

$$[dd.2]_a = [dd] - \frac{[bd]^2}{bb} - \frac{([bc][bd]/[bb] - [cd])^2}{[cc.1]_a}.$$

These, by omitting all factors containing d, reduce to the same expressions as above derived for the case of three unknowns, x, y, z.

157. *Problems.*

1. Three observations on a single quantity furnish the observation equations $3x = 2.18$, $2x = 1.44$, $4x = 2.90$. Find the most probable value of x and its probable error.

2. Observations made in a deep well near Paris on the temperature at different depths below the surface of the earth gave the following results, t being the temperature corresponding to the depth d:

For $d =$ 28 meters,	$t = 11^\circ.71$ C.,	for $d =$ 298 meters,	$t = 22^\circ.20$ C.
$d =$ 66	$t = 12.90$	$d = 400$	$t = 23.75$
$d = 173$	$t = 16.40$	$d = 505$	$t = 26.43$
$d = 248$	$t = 20.00$	$d = 548$	$t = 27.70$

Assume the temperature at the surface ($d = 0$), to be the annual mean $t_0 = 10^\circ.60$, and also that the law of variation of t with d is given by

$$t = t_0 + Sd + Td^2.$$

State the observation equations, form the normal equations, solve them by the method of Art. 154, and show that the most probable values of S and T are $+0.04153 \pm 0.00165$ and $-0.00001929 \pm 0.00000356$.

3. Given the three normal equations,

$$6.649x + 2.041y + 2.941z - 1.00 = 0,$$
$$2.041x + 4.249y + 0.926z - 1.35 = 0,$$
$$2.941x + 0.926y + 5.382z - 0.92 = 0.$$

Form the sums $[as]$, $[bs]$, $[cs]$, $[ms]$, and then solve the equations by the use of logarithms.

4. At a station O angles were measured as follows between the five stations, A, B, C, D, E:

$AOB =$	15°	37′	32″.67,	weight 6,
$AOC =$	45	20	47.34,	weight 4,
$AOD =$	156	23	28.76,	weight 8,
$AOE =$	268	44	19.84,	weight 3,
$BOC =$	29	43	13.56,	weight 2,
$BOD =$	140	45	57.13,	weight 6,
$BOE =$	253	06	45.03,	weight 1,
$COD =$	111	02	42.86,	weight 4,
$COE =$	223	23	30.94,	weight 8,
$DOE =$	112	20	49.32,	weight 2.

Let x, y, z, w, be the most probable corrections to the observed values of AOB, AOC, AOD, AOE. State the observation equations, form and solve the normal equations, show that $w = +0''.51$, $z = +2''.36$, etc., and that the adjusted values of the observed angles are $AOB = 15° \; 37' \; 32''.12, \ldots . \; DOE = 112° \; 20' \; 49''.15$. Also show that the weight of the adjusted value of AOE is 8.9 and that its probable error is $\pm 0''.91$.

5. Solve the following normal equations:

$$+380.95x + 142.86y - 16.88z - 68.63w - 36.67 = 0,$$
$$+142.86x + 428.57y + 30.96z + 95.00 = 0,$$
$$-16.88x + 30.96y + 208.34z - 6.08w - 121.51 = 0,$$
$$-68.63x - 6.08z + 80.54w + 16.34 = 0,$$

and show that the value of x is $+0.275$.

CHAPTER XI.

APPENDIX AND TABLES.

158. The elementary principles and applications of the Method of Least Squares have now been given and exemplified. It remains to note a few points that have not found a place in the preceding chapters, to present some remarks on the history and literature of the subject, and to give several tables that will be useful in abridging computations.

Observations Involving Non-Linear Equations.

159. It has been thus far assumed that the observations can be represented by equations of the first degree. If this is not the case, and higher equations are involved, they can be reduced to linear ones by the following method:

Let the q quantities to be determined be represented by $z_1, z_2 \ldots z_q$, and the n measured quantities by $M_1, M_2 \ldots M_n$, and let the n observation equations be of the form

$$\phi = f(z_1, z_2 \ldots z_q) = M.$$

Now, let approximate values of the unknown quantities be found, either by trial, or by the solution of a sufficient number of equations, and let them be denoted by $Z_1, Z_2 \ldots Z_q$. Let

z_1', $z_2' \ldots z_q'$ be the most probable corrections to these approximate values; so that

$$z_1 = Z_1 + z_1', \quad z_2 = Z_2 + z_2' \ldots z_q = Z_q + z_q'.$$

If, now, each of the functions ϕ be developed by Taylor's theorem, and the products and higher powers of the corrections be neglected, there will be n expressions of the form

$$\phi = f(Z_1, Z_2 \ldots Z_q) - M + \frac{d\phi}{dz_1} z_1' + \frac{d\phi}{dz_2} z_2' + \ldots + \frac{d\phi}{dz_q} z_q' = 0.$$

Here the terms $f(Z_1, Z_2 \ldots Z_q)$ are known, and may be designated by $N_1, N_2 \ldots N_q$; so that the n expressions reduce to the form

$$\frac{d\phi}{dz_1} z_1' + \frac{d\phi}{dz_2} z_2' + \ldots + \frac{d\phi}{dz_q} z_q' = M - N,$$

where the differential co-efficients are to be found by differentiating each of the observation equations with reference to each variable, and then substituting the approximate values $Z_1, Z_2 \ldots Z_q$, for $z_1, z_2 \ldots z_q$. Denoting them, then, by a, b, c, etc., the n equations are of the form

$$az_1 + bz_2 + cz_3 + \ldots + lz_q = M - N,$$

in which all the letters except $z_1, z_2 \ldots z_q$, denote known quantities. These n equations are exactly like the observation equations (10) or (12), and from them the normal equations are formed, whose solution furnishes the most probable values of the corrections.

If non-linear conditional equations are given, it is also necessary to find approximate values for the unknown quantities, and assume a system of corrections. Then the functional conditional equations may be developed as above, and reduced to

linear equations containing the corrections as unknowns, which may be treated by the method of correlatives, and the most probable system of corrections determined, which, applied to the approximate values, will give the adjusted results. If these do not satisfy the original equations with sufficient accuracy, a new system of corrections should be assumed, and the process be again repeated.

Certain expressions, like that in Art. 111, may be reduced to the linear form by the help of logarithms; and, when this is possible, it will be found a more convenient method of treatment than the development by Taylor's theorem.

160. As an illustration of the method, let M be the numbei of millions of people under the age of m years, and let it be required to find the most probable value of z in the empirical formula

$$\phi = 50.16 \sin m(0.996)^m z = M,$$

which is supposed to give the relation between M and m for the population of the United States in 1880. The data are nine values of M, from the census compendium, given in the second column of the table below.

The first step is to find by trial that $1^\circ.55$ is an approximate value of the angle z. The second is to compute nine values of the expression

$$50.16 \sin m(0.996)^m 1^\circ.55 = N,$$

corresponding to the nine given values of m: these are put in the third column of the table. In the fourth column are the differences $M - N$ between the observed and computed values. The fifth column contains the values of the derivative

$$\frac{d\phi}{dz} = 50.16 m(0.996)^m \cos m(0.996)^m 1^\circ.55,$$

corresponding to the nine values of m.

m.	M.	N.	$M-N$.	$\frac{d\phi}{dz}$.
10	13.39	12.90	+ 0.49	466
20	24.12	24.06	+ 0.06	814
30	33.29	33.07	+ 0.22	1004
40	39.66	40.08	− 0.42	1032
50	44.22	44.98	− 0.76	913
60	47.33	48.10	− 0.77	674
70	49.16	49.74	− 0.58	349
80	49.94	50.14	− 0.20	29
90	50.14	49.67	+ 0.47	447

Let z' be the most probable correction to the assumed value of z. Then the last two columns furnish the nine observation equations

$$466z' = +0.49,$$
$$814z' = +0.06, \quad \text{etc.}$$

From these the single normal equation is formed, and its solution gives

$$z' = -0^a.00025 = -0^\circ.014,$$

and hence the most probable value of z is

$$z = 1^\circ.55 - 0^\circ.014 = 1^\circ.536;$$

so that the empirical formula may be written

$$M = 50.16 \sin m (0.996)^m 1^\circ.536.$$

When z is $1^\circ.55$, the sum of the squares of the residuals is about 2.25; and, when z is $1^\circ.536$, that sum is about 1.67; so that

the precision of the formula has not been greatly increased by the variation in the angle z. By slightly increasing the number 0.996, the formula can be made to more closely agree with the observations.

Mean Error and Probable Error.

161. The probable error is an error of such a value that any given error is as likely to exceed it as be less than it, and it hence seems to be the quantity that would most naturally be selected for indicating the precision of observations. But there is another error very commonly employed for the same purpose called the "mean error," whose definition is, the error whose square is the mean of the squares of all the errors. Hence the mean error is, for direct observations, the square root of the quantity $\frac{\Sigma x^2}{n}$, or, in terms of the residuals, the square root of the quantity $\frac{\Sigma v^2}{n-1}$. In general, then, the mean error can be determined from the formulas for probable error by changing the co-efficient 0.6745 into unity. If m be the mean, and r the probable error, the relation between them is

$$m = \frac{r}{0.6745} = 1.4826r.$$

In the annexed figure, OP indicates the probable error, and OM the mean error. It is seen, by Art. 29, that M is the abscissa of the point of inflection of the probability curve.

In Table II, the value of the integral for the argument $1.4826r$ is 0.6826. Hence 0.6826 is the probability that an error is less than the mean error, or in 1,000 errors there should be 683 less than m. It is a fair wager of 1 to 1 that an error taken at random is less than the probable error; but it is a fair wager of 683 to 317, or about 2.15 to 1, that it is less than the mean error.

The mean error is generally used in German books. In this country the probable error is commonly employed; and, being the most natural unit of comparison, it is certainly to be desired that it alone should be used, and the mean error be discarded.

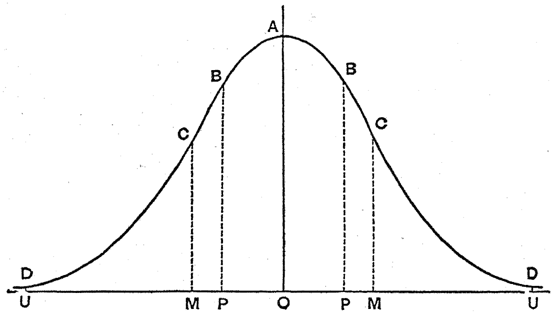


Fig. 14.

162. Instead of the mean or probable error, a quantity called the "huge error" might be employed to indicate the precision of measurements. The huge error is defined to be an error of such a magnitude that 999 errors out of 1,000 are less than it, and only 1 greater; or, in other words, that the probability of an error being less than it, is 0.999. If u be the huge error, the relation of u to r is found from Table II. For $P = 0.999$, the argument $\frac{u}{r}$ is 4.9: hence

$$u = 4.9r.$$

Accordingly, all formulas for probable error may be changed into those for huge error by writing 3.3 in place of 0.6745. For instance, the huge error of a single direct observation is given by

$$u = 3.3\sqrt{\frac{\Sigma v^2}{n-1}}.$$

In Fig. 14 the abscissa OU represents the huge error, and the area $UDADU$ is 0.999 of the total area.

Uncertainty of the Probable Error.

163. The value of the probable error r, deduced in Art. 67, is the best attainable value, or rather the most probable value. The inquiry is now to be made as to what are the probable limits of this value of r, or what is the probable error of the probable error. Or, if the probable error r be written in the form

$$r(1 \pm u),$$

the number u is the uncertainty of the probable error, that is, it as likely that the value formed for r lies between the limits $r(1 - u)$ and $r(1 + u)$ as that it lies outside those limits. Thus ur may properly be called the probable error of the probable error r.

164. A series of observations having been made, all having the same measure of precision h, the sum of the squares of the errors is a constant, while the probability of any value h is, by Art. 65,

$$P' = h^n(dx)^n \pi^{-\frac{1}{2}n} e^{-h^2\Sigma x_2},$$

and the value of h which renders this a maximum is the most probable value of h. Now let $h + uh$ be a value greater than this probable value; then

$$P'' = (h + uh)^n(dx)^n \pi^{-\frac{1}{2}n} e^{-(h + uh)^n\Sigma x^2}$$

is the probability of the value $h + uh$. The ratio of these probabilities is

$$\frac{P''}{P'} = (1 + u)^n e^{-(2u + u^2)h^2\Sigma x^2},$$

and taking the logarithms of both sides of this equation,

$$\log \frac{P''}{P'} = n \log(1 + u) - (2u + u^2)h^2\Sigma x^2;$$

also replacing $\log(1+u)$ by $u - \frac{1}{2}u^2$, the terms involving higher powers of u being omitted, there results

$$\log\frac{P''}{P'} = (n - 2h^2\Sigma x^2)u - (\tfrac{1}{2}n + h^2\Sigma x^2)u^2.$$

The value of h deduced in Art. 65 causes the co-efficient of u to become zero, whence

$$\log\frac{P''}{P'} = -nu^2 \quad\text{and}\quad \frac{P''}{P'} = e^{-nu^2}.$$

Thus the probability of the variation uh in the value of h is expressed by the function

$$P'' = ce^{-nu^2},$$

which is of the same form as the law of accidental error.

The probability that u is less than any assigned limit is therefore, as in Art. 32, expressed by the integral

$$2\sqrt{\frac{n}{\pi}}\int_0^u e^{-nu^2}du = \frac{2}{\sqrt{\pi}}\int_0^t e^{-t^2}dt,$$

and the value of this integral is $\frac{1}{2}$, as in Art. 61, when

$$t = u\sqrt{n} = 0.4769.$$

Consequently the probable error of the measure of precision h is

$$u = \frac{0.4769}{\sqrt{n}},$$

and hence the probable limits of h are

$$h\left(1 + \frac{0.4769}{\sqrt{n}}\right) \quad\text{and}\quad h\left(1 - \frac{0.4769}{\sqrt{n}}\right).$$

Thus the uncertainty in the probable value of h has been found.

Now, since $hr = 0.4769$, the uncertainty in the value of r is

the same as that in the value of h. The probable limits of the probable value of the probable error r are, therefore,

$$r\left(1 - \frac{0.4769}{\sqrt{n}}\right) \quad \text{and} \quad r\left(1 + \frac{0.4769}{\sqrt{n}}\right),$$

and the uncertainty in r decreases directly as the square root of the number of observations. Thus for four observations the uncertainty in r is 24 per cent of its value, for 16 observations it is 12 per cent of its value.

165. The above supposes that the probable error is computed by the sums of the squares of the residuals according to formulas (20) and (21). If, however, formulas (35) and (36) be employed, using the sum of the residuals only, then a similar investigation will show that

$$r\left(1 - \frac{0.5096}{\sqrt{n}}\right) \quad \text{and} \quad r\left(1 + \frac{0.5096}{\sqrt{n}}\right),$$

are the probable limits of the probable error r. Here the uncertainty is greater than in the former case, 114 observations being necessary to give the same uncertainty in the probable error as 100 observations give when (20) and (21) are used.

It may be noted, finally, that some writers state the above expressions for the uncertainty so that $\sqrt{n-1}$ appears in the denominator instead of $\sqrt{n}$.

The Median.

166. When an odd number of direct measurements are made on a single quantity, the middle one in the order of numerical magnitude is called the median. Thus, if the results of nine direct observations are

103, 104, 105, 106, 106, 107, 108, 110, 111,

the fifth one, counting from either end, is 106, which is the median.

If the number of observations be even, the median is the mean of the two middle ones in the order of magnitude. Thus, if to the above observations there be added 112, then the median is $\frac{1}{2}(106 + 107) = 106\frac{1}{2}$. In the first case the arithmetical mean is $106\frac{2}{3}$ and in the second case it is 107.2. The median in general differs from the arithmetical mean.

When observations are weighted these weights are to be used in counting off the large and small observations until the middle one or the two middle ones are found, and then an interpolation is made to find the median. For example, let

$$\text{Observation} = 1, \quad 2, \quad 3, \quad 4, \quad 5,$$
$$\text{Weight} = 2, \quad 5, \quad 16, \quad 10, \quad 7.$$

Here the sum of the weights is 40, which may be taken as the total number of direct observations, and the median plainly lies between $2\frac{1}{2}$ and $3\frac{1}{2}$. Seven observations are less than $2\frac{1}{2}$ and seventeen are greater than $3\frac{1}{2}$; thus sixteen observations may be said to lie between $2\frac{1}{2}$ and $3\frac{1}{2}$, and this interval is to be divided in the ratio of $20 - 7$ to $20 - 17$. The median hence is $2\frac{1}{2} + \frac{13}{16} = 3\frac{5}{16}$, or again $3\frac{1}{2} - \frac{3}{16} = 3\frac{5}{16}$.

167. The probable error of a single observation is to be found by counting off one-fourth of the residual errors from both ends, and if these are not equal their mean may be taken. Thus, for the following case where the median is 33,

$$\begin{array}{lcccccccc} \text{Observation} = & 31, & 32, & 32, & 33, & 33, & 34, & 35, & 36, \\ \text{Residual} \quad = & 2, & 1, & 1, & 0, & 0, & 1, & 2, & 3, \end{array}$$

the probable error found by counting off two residuals from the left is 1.0, while by counting off two from the right it is 1 5, the mean of these being 1.25, and then

$$r_0 = \frac{1.25}{\sqrt{8}} = 0.44,$$

is the probable error of the median itself.

The median was first suggested by Galton in 1875* as a convenient method of obtaining a mean without the necessity of making many measurements. For example, if it were desired to obtain the mean height of the boys in a school they might be arranged in a row in the order of height and then the measurement of the middle boy would give the median. Further, if the probable variation in height were required it would be only necessary to measure the two boys standing at the quarter points of the line, and then subtract the mean of their heights from the median. This gives the probable error of a single height, and by dividing it by the square root of the number of boys the probable error of the median height is obtained.

The median, when obtained by the process indicated by Galton, may be regarded as a representative value of the mean quantity which is desired. But when all the individual measures are actually taken, the arithmetical mean and not the median is the most probable value, provided that the law of variation is the same as the law of facility of accidental error. To take the median in the latter case, for the sake of avoiding computation, can only be justified when the observations are rough ones, and then the median itself is liable to differ considerably from the arithmetical mean. The use of the median, except in the manner indicated by Galton, does not seem warranted in cases of symmetric probability.

The uncertainty of the probable error of the median is greater than that of the arithmetical mean, 217 observations being necessary in the former case to give the same uncertainty as 100 observations give in the latter case.†

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* Statistics by Intercomparison, Philosophical Magazine, vol. xlix, p. 33.

† See Gauss, Werke, vol. iv, pp. 109–117. See also Scripture, On mean values from direct measurements, in Studies from Yale Psychological Laboratory, 1894, vol. ii, pp. 1–39.

History and Literature.

168. The average or arithmetical mean has, from the earliest times, been employed for the determination of the most probable value of a quantity observed several times with equal care. From this arises so naturally the idea of weights and of the weighted mean, that undoubtedly both were in use long before any attempt was made to deduce general laws based upon mathematical principles. About the year 1750 certain indirect observations in astronomy led to observation equations, and the question as to the proper manner of their solution arose. Boscovich in Italy, Mayer and Lambert in Germany, Laplace in France, Euler in Russia, and Simpson in England, proposed different methods for the solution of such cases, discussed the reasons for the arithmetical mean, and endeavored to determine the law of facility of error. Simpson, in 1757, was the first to state the axiom that positive and negative errors are equally probable; and Laplace, in 1774, was the first to apply the principles of probability to the discussion of errors of observations. Laplace's method for finding the values of q unknown quantities from n observation equations consisted in imposing the conditions that the algebraic sum of the residuals should be zero, and that their sum, all taken with the positive sign, should be a minimum. By introducing these conditions, he was able to reduce the n equations to q, from which the q unknowns were determined. This method he applied to the deduction of the shape of the earth from measurements of arcs of meridians, and also from pendulum observations.

The honor of the first statement of the principle of Least Squares is due to Legendre, who in 1805 proposed it as an advantageous and convenient method of adjusting observations. He called it "Méthode des moindres quarrés," showed that the rule of the arithmetical mean is a particular case of the

general principle, deduced the method of normal equations, and gave examples of its application to the determination of the orbit of a comet and to the form of a meridian section of the earth. Although Legendre gave no demonstration that the results thus determined were the most probable or best, yet his remarks indicated that he recognized the advantages of the method in equilibrating the errors.

The first deduction of the law of probability of error was given in 1808 by Adrain, in "The Analyst," a periodical published by him at Philadelphia. From this law he showed that the arithmetical mean followed, and that the most probable position of an observed point in space is the centre of gravity of all the given points. He also applied it to the discussion of two practical problems in surveying and navigation.

In 1809 Gauss deduced the law of probability of error as in Arts 27, 28, and from it gave a full development of the method. To Gauss is due the algorithm of the method, the determination of weights from normal equations, the investigation of the precision of results, the method of correlatives for conditional observations, and numerous practical applications. Few branches of science owe so large a proportion of subject-matter to the labors of one man.

The method thus thoroughly established spread among astronomers with rapidity. The theory was subjected during the following fifty years to rigid analysis by Encke, Gauss, Hagen, Ivory, and Laplace, while the labors of Bessel, Gerling, Hansen, and Puissant, developed its practical applications to astronomical and geodetical observations. During the period since 1850, the literature of the subject has been greatly extended. The writings of Airy and De Morgan in England, of Liagre and Quetelet in Belgium, of Bienaymé in France, of Schiaparelli in Italy, of Andrä in Denmark, of Helmert and Jordan in Germany, of Chauvenet and Schott in the United States, have

brought the science to a high degree of perfection in all its branches, and have caused it to be universally adopted by scientific men as the only proper method for the discussion of observations.

169. In 1877 the author published, in the "Transactions of the Connecticut Academy," a list of writings relating to the Method of Least Squares and the theory of the accidental errors of observation, which comprised 408 titles. These were classified as 313 memoirs, 72 books, and 23 parts of books. They were written by 193 authors, 127 of whom produced only one book or paper each. The date of publication of the earliest is 1722. From that time to 1805, the year of Legendre's announcement of the principle of Least Squares, there are 22 titles; since 1805 there is a continual yearly increase in the number; thus:

From 1805 to 1814 inclusive, there are 18 titles.
From 1815 to 1824 inclusive, there are 30 titles.
From 1825 to 1834 inclusive, there are 32 titles.
From 1835 to 1844 inclusive, there are 45 titles.
From 1845 to 1854 inclusive, there are 63 titles.
From 1855 to 1864 inclusive, there are 71 titles.
From 1865 to 1874 inclusive, there are 95 titles.

The books and memoirs are in eight languages; and, classified according to the place of publication, they fall under twelve countries. It may be interesting to note the number belonging to each; thus:

Countries.		Countries.	
Germany	153	Austria	10
France	78	Switzerland	9
Great Britain	56	Holland	7
United States	34	Sweden	7
Belgium	19	Denmark	5
Russia	16		
Italy	14	Total	408

Languages.		Languages.	
German	167	Dutch	7
French	110	Danish	5
English	90	Swedish	4
Latin	16		
Italian	9	Total	408

The titles of papers and books issued since 1876 may be mostly found in the excellent publication "Jahrbuch über die Fortschritte der Mathematik."*

Constant Numbers.

170. In the preceding pages the constant numbers entering the formulas for probable error have been stated only to four decimal places, which is entirely sufficient for any practical computation. As a matter of mathematical interest, however, they are here given to seven decimals, together with a few other related constants and their common logarithms.

Symbol.	Constant.	Logarithm.
hr	0.4769363	$\bar{1}.6784604$
$hr\sqrt{2}$	0.6744897	$\bar{1}.8289754$
$hr\sqrt{\pi}$	0.8453476	$\bar{1}.9270353$
$\sqrt{2}$	1.4142136	0.1505150
π	3.1415927	0.4971499
$\sqrt{\pi}$	1.7724539	0.2485749
$\pi^{-\frac{1}{2}}$	0.5641896	$\bar{1}.7514251$
e	2.7182818	0.4342945
Mod.	0.4342945	$\bar{1}.6377843$

* Gore's Bibliography of Geodesy, published in the U. S. Coast and Geodetic Survey Report for 1887, will be found excellent on the subject of the method of least squares.

Answers to Problems; and Notes.

171. Below are given answers to a number of the problems stated in the text and hints concerning the solution of others, together with explanatory notes upon some of the more difficult points in the theory of the subject.

Article 16.—Problem 2: $\frac{1}{22}$. Problem 3: 0.9308 by the use of Table V. Problem 5: Find the probability of a hundred heads in a single throw of a hundred coins, and multiply this by the number of inhabitants and the number of seconds to find the probability of the occurrence under the given data. Problem 6: The probability that the nickel is in the first purse is $\frac{10}{19}$.

Article 26.—The equation at the foot of page 20 may be written in the form

$$\frac{y - y'}{y} = \frac{2(\varDelta x - x)}{(m + 2)\varDelta x - x},$$

and, in passing to the limit, $y - y'$ is infinitely small compared to y, and $\varDelta x$ vanishes with respect to x. Hence in the second member $2x$ is infinitely small compared to the denominator, and accordingly x vanishes with respect to $(m + 2)\varDelta x$.

Article 37.—Problem 2: see Fig. 2 and Fig. 6. Problem 3: Show this by the principle of sufficient reason. Problem 5: because k depends upon h and h is different in the two cases. Problem 6: From formula (2) an expression for $\pi^{\frac{1}{2}}$ is found, then h, dx, x, and y are derived by observation and π is computed; thus for the case of Article 33 the probability of the error $3''.5$ may be roughly taken as that of the occurrence between the limits $3''.0$ and $4''.0$, so that the observed value of y is $\frac{4}{100}$, and as dx is $1''.0$, there results

$$\pi^{\frac{1}{2}} = \frac{hdx}{ye^{h^2x^2}} = \frac{100 \times 1}{242.236 \times 11.57} = 2.11;$$

whence $\pi = 4.48$, a rude result indeed, but by increasing the number of observations and decreasing the interval between the successive errors a closer accordance may be secured.

Article 59.—Problem 2: *N.* 2′.4 *E.* Problem 3: $z_1 = -0.19$, $z_2 = +0.14$, $z_3 = +0.05$, etc. Problem 4: $\frac{1}{13}d$ to A, $\frac{2}{13}$ to B, and $\frac{10}{13}$ to C, the greater the weight the less being the amount of correction.

Article 67.—The reason why Σpx^2y is the same as $\dfrac{\Sigma px}{n}$ is sometimes not clear to students. If each term such as $p_1x_1^2$ occurs ny_1 times in n observations, then

$$p_1x_1^2.ny_1 + p_2x_2^2.ny_2 + \text{etc.} = \Sigma px^2,$$

or

$$n(p_1x_1^2y_1 + p_2x_2y_2 + \text{etc.}) = \Sigma px^2;$$

whence, dividing by n, follows the statement as given.

Article 89.—Problem 1: 0″.408. Problems 3 and 4: The combination of observations differing widely in precision, as in these examples, is not always safe in practice, because of the constant errors which are liable to affect the less precise series, so that the practical weight of the more precise series is often greater than that derived from the probable errors. Problem 6: It should be inferred that a constant source of error exists.

Article 98.—Problem 2: 0.000137, which occurs when A is 135 degrees. Problem 4: 0.005. Problem 6: The probable error of the mean of the three readings is $\dfrac{0.001}{\sqrt{3}}$, and that of the difference of level of two stations is this multiplied by $\sqrt{2}$; then for the 130 stations there are 129 differences of level, and the probable error of the final result is 0.0093 feet.

Article 107.—The proof of this method may be made in the following manner: Let x_1' and y_1' be the adjusted values of the observations x_1 and y_1, so that the residual errors are

$x_1' - x_1$ and $y_1' - y_1$. Then the most probable values of S and T are to be found from the condition

$$\Sigma p(x_1' - x_1)^2 + \Sigma(y_1' - y_1)^2 = \text{a minimum}.$$

The adjusted points all lie upon the line whose equation is $y = Sx + T$. Now let a second line be drawn through the observed point whose co-ordinates are x_1 and y_1, and the adjusted point whose co-ordinates are x_1' and y_1'; its equation is $y - y_1 = S'(x - x_1)$. By combining this with the equation of the required line the values of the residual errors are deduced, whence the above condition becomes

$$\frac{p + S'^2}{(S' - S)^2}\Sigma(Sx + T - y)^2 = \text{a minimum}.$$

This is to be made a minimum for S', S, and T separately. Taking the derivative with respect to S' and equating to zero there is found $S'S + p = 0$, which gives the inclination S' in terms of S. Again, differentiating with respect to S and T there are deduced two equations in S and T, namely,

$$S^2\Sigma xy - S^2T\Sigma x - S\Sigma y^2 + 2ST\Sigma y - nST^2 + pS\Sigma x^2 - p\Sigma xy + pT\Sigma x = 0,$$
$$S\Sigma x + nT - \Sigma y = 0,$$

and the solution of these gives values for S and T which agree with the results stated in the text.

Article 112.—Problem 6: Let p represent the population in millions and x the number of decades since 1800. Then using the ten censuses from 1790 to 1880, there is found

$$p = 4.97 + 0.873x + 0.581x^2,$$

which gives 59 890 000 for 1890, while the actual enumeration was 62 870 000. Again, taking the seven censuses from 1820 to 1880, there is found

$$p = 7.29 - 0.280x + 0.689x^2,$$

which gives 60 579 000 for 1890, an accordance more satisfactory. The sum of the squares of the residual errors for the

latter formula is 1.35, while for the same seven census years the former gives 3.34.

Article 113.—Whether observations shall be independent or conditioned depends in general upon the selection of the unknown quantities whose values are to be determined. Thus if AOB, BOC, and AOC are angles measured at a station O, the observation equations are independent if x and y be put for two of these angles. But if x, y, and z are taken as the three quantities, these are conditioned by the necessary relation that the sum of two of them is equal to the third.

Article 126.—Problem 1: Refer to problem 4 of Article 59. Problem 2: $x = 93° 48' 14''.99$, $y = 51° 54' 59''.84$, $z = 34° 16' 49''.22$. Problem 5: This was the problem proposed by Patterson in 1808, and by whose discussion Adrain was led to the discovery of the principle of least squares.

Article 144.—The arithmetical mean of more than two observations is, in strictness, the most probable value only when the results of the measurements are unknown. If the mind knows the values of the measurements, it instinctively assigns greater reliability to some than to others, and hence the weights are not equal. For example, let $M_1 = 40$, $M_2 = 51$, $M_3 = 52$ be three observations of the same quantity: it is reasonable to suppose that M_1 is of less reliability than the others, while the method of the mean assigns it the same weight. Theory has not been able to determine what theoretical weights should be assigned in a case like this, but probably an approach to them might be secured by taking the reciprocal of $(M_1 - M_2)^2 + (M_1 - M_3)^2$ as the weight of M_1, the reciprocal of $(M_2 - M_1)^2 + (M_2 - M_3)^2$ as the weight of M_2, and that of $(M_3 - M_1)^2 + (M_3 - M_2)^2$ as the weight of M_3. For the above numerical example this gives $\frac{1}{265}$ as the weight of 40, $\frac{1}{122}$ as the weight of 51, and $\frac{1}{145}$ as the weight of 52, from which results the general mean $z = 49.18$, whereas the arithmetical mean is 47.67.

Description of the Tables.

172. Tables I and II give values of the probability integral (4); the first for the argument hx, and the second for the argument $\frac{hx}{0.4769}$, or $\frac{x}{r}$. In both cases the arrangement is like that of logarithmic tables, and needs no explanation. The use of Table I is illustrated in Arts. 32 and 33, and that of Table II in Art. 128. These tables were first given by Encke in 1832, and were computed by him from a table of the values of $\int e^{-t^2}dt$, which was published by Kramp in 1799.

Tables III and IV give values of the co-efficients which occur in the formulas for probable error for values of n. Table III applies to the usual formulas (20) and (21), and its use is illustrated in Art. 82. Table IV applies to the shorter formulas (35) and (36), and its use is illustrated in Art. 84. These tables were computed by Wright, and first published in "The Analyst" for 1882, vol. ix, p. 74.

Table V gives four-place logarithms of numbers, and Table VI gives four-place squares of numbers. The latter will be found very useful for obtaining the squares of residuals. It may be also used in forming the co-efficients in normal equations, and for other purposes. For instance, the co-efficient $[ab]$ may be written

$$[ab] = \tfrac{1}{2}([(a+b)^2] - [a^2] - [b^2]),$$

and the sums $[a^2]$, $[b^2]$, and $[(a+b)^2]$ may be easily formed with the help of the table of squares. This method has the advantage that no attention need be paid to the signs of a and b, except in forming the sums $a+b$.

Table VII is to be used in discussing doubtful observations by Chauvenet's criterion, and its use is explained in Art. 130.

Table VIII gives the squares of reciprocals of numbers from 0.0 to 9.0, and may be used in the computation of weights from probable errors.

TABLE I.

Values of the Probability Integral $\frac{2}{\sqrt{\pi}}\int_0^t e^{-t^2}dt$ for Argument t or hx.

hx.	0	1	2	3	4	5	6	7	8	9	Diff.
0.0	0.0000	0.0113	0.0226	0.0338	0.0451	0.0564	0.0676	0.0789	0.0901	0.1013	113
0.1	1125	1236	1348	1459	1569	1680	1790	1900	2009	2118	110
0.2	2227	2335	2443	2550	2657	2763	2869	2974	3079	3183	106
0.3	3286	3389	3491	3593	3694	3794	3893	3992	4090	4187	100
0.4	4284	4380	4475	4569	4662	4755	4847	4937	5027	5117	92
0.5	0.5205	0.5292	0.5379	0.5465	0.5549	0.5633	0.5716	0.5798	0.5879	0.5959	83
0.6	6039	6117	6194	6270	6346	6420	6494	6566	6638	6708	74
0.7	6778	6847	6914	6981	7047	7112	7175	7238	7300	7361	64
0.8	7421	7480	7538	7595	7651	7707	7761	7814	7867	7918	55
0.9	7969	8019	8068	8116	8163	8209	8254	8299	8342	8385	45
1.0	0.8427	0.8468	0.8508	0.8548	0.8586	0.8624	0.8661	0.8698	0.8733	0.8768	37
1.1	8802	8835	8868	8900	8931	8961	8991	9020	9048	9076	30
1.2	9103	9130	9155	9181	9205	9229	9252	9275	9297	9319	23
1.3	9340	9361	9381	9400	9419	9438	9456	9473	9490	9507	18
1.4	9523	9539	9554	9569	9583	9597	9611	9624	9637	9649	14
1.5	0.9661	0.9673	0.9684	0.9695	0.9706	0.9716	0.9726	0.9736	0.9745	0.9755	10
1.6	9763	9772	9780	9788	9796	9804	9811	9818	9825	9832	7
1.7	9838	9844	9850	9856	9861	9867	9872	9877	9882	9886	5
1.8	9891	9895	9899	9903	9907	9911	9915	9918	9922	9925	4
1.9	9928	9931	9934	9937	9939	9942	9944	9947	9949	9951	3
2.0	0.9953	0.9955	0.9957	0.9959	0.9961	0.9963	0.9964	0.9966	0.9967	0.9969	2
2.1	9970	9972	9973	9974	9975	9976	9977	9979	9980	9980	1
2.2	9981	9982	9983	9984	9985	9985	9986	9987	9987	9988	1
2.3	9989	9989	9990	9990	9991	9991	9992	9992	9992	9993	
2.4	9993	9993	9994	9994	9994	9995	9995	9995	9995	9996	
2.	0.9953	0.9970	0.9981	0.9989	0.9993	0.9996	0.9998	0.9999	0.9999	0.9999	
∞	1.0000										
hx.	0	1	2	3	4	5	6	7	8	9	Diff.

TABLE II.

Values of the Probability Integral $\frac{2}{\sqrt{\pi}}\int_0^t e^{-t^2}dt$ for Argument $\frac{t}{0.4769}$ or $\frac{x}{r}$

$\frac{x}{r}$	0	1	2	3	4	5	6	7	8	9	Diff.
0.0	0.0000	0.0054	0.0108	0.0161	0.0215	0.0269	0.0323	0.0377	0.0430	0.0484	54
0.1	0538	0591	0645	0699	0752	0806	0859	0913	0966	1020	54
0.2	1073	1126	1180	1233	1286	1339	1392	1445	1498	1551	53
0.3	1603	1656	1709	1761	1814	1866	1918	1971	2023	2075	52
0.4	2127	2179	2230	2282	2334	2385	2436	2488	2539	2590	51
0.5	0.2641	0.2691	0.2742	0.2793	0.2843	0.2893	0.2944	0.2994	0.3043	0.3093	50
0.6	3143	3192	3242	3291	3340	3389	3438	3487	3535	3583	49
0.7	3632	3680	3728	3775	3823	3870	3918	3965	4012	4059	46
0.8	4105	4152	4198	4244	4290	4336	4381	4427	4472	4517	45
0.9	4562	4606	4651	4695	4739	4783	4827	4860	4914	4957	43
1.0	0.5000	0.5043	0 5085	0.5128	0.5170	0.5212	0.5254	0.5295	0.5337	0.5378	41
1.1	5419	5460	5500	5540	5581	5620	5660	5700	5739	5778	39
1.2	5817	5856	5894	5932	5970	6008	6046	6083	6120	6157	37
1.3	6194	6231	6267	6303	6339	6375	6410	6445	6480	6515	35
1.4	6550	6584	6618	6652	6686	6719	6753	6786	6818	6851	32
1.5	0.6883	0.6915	0.6947	0.6979	0.7011	0.7042	0.7073	0.7104	0.7134	0.7165	30
1.6	7195	7225	7255	7284	7313	7342	7371	7400	7428	7457	28
1.7	7485	7512	7540	7567	7594	7621	7648	7675	7701	7727	26
1.8	7753	7778	7804	7829	7854	7879	7904	7928	7952	7976	24
1.9	8000	8023	8047	8070	8093	8116	8138	8161	8183	8205	22
2.0	0.8227	0.8248	0.8270	0.8291	0.8312	0.8332	0.8353	0.8373	0.8394	0.8414	19
2.1	8433	8453	8473	8492	8511	8530	8549	8567	8585	8604	18
2.2	8622	8639	8657	8674	8692	8709	8726	8742	8759	8775	17
2.3	8792	8808	8824	8840	8855	8870	8886	8901	8916	8930	15
2.4	8945	8960	8974	8988	9002	9016	9029	9043	9056	9069	13
2.5	0.9082	0.9095	0.9108	0.9121	0.9133	0.9146	0.9158	0.9170	0.9182	0.9193	12
2.6	9205	9217	9228	9239	9250	9261	9272	9283	9293	9304	10
2.7	9314	9324	9334	9344	9354	9364	9373	9383	9392	9401	9
2.8	9410	9419	9428	9437	9446	9454	9463	9471	9479	9487	8
2.9	9495	9503	9511	9519	9526	9534	9541	9548	9556	9563	7
3.0	0.9570	0.9577	0.9583	0.9590	0.9597	0.9603	0.9610	0.9616	0.9622	0.9629	6
3.1	9635	9641	9647	9652	9658	9664	9669	9675	9680	9686	5
3.2	9691	9696	9701	9706	9711	9716	9721	9726	9731	9735	5
3.3	9740	9744	9749	9753	9757	9761	9766	9770	9774	9778	4
3.4	9782	9786	9789	9793	9797	9800	9804	9807	9811	9814	4
3.	0.9570	0.9635	0.9691	0.9740	0.9782	0.9818	0.9848	0.9874	0.9896	0.9915	
4.	9930	9943	9954	9963	9970	9976	9981	9985	9988	9990	
5.	9993	9994	9996	9997	9997	9998	9998	9999	9999	9999	
∞	1.0000										
$\frac{x}{r}$	0	1	2	3	4	5	6	7	8	9	Diff.

TABLE III.

For Computing Probable Errors by Formulas (20) and (21).

n.	$\frac{0.6745}{\sqrt{n-1}}$.	$\frac{0.6745}{\sqrt{n(n-1)}}$.	n.	$\frac{0.6745}{\sqrt{n-1}}$.	$\frac{0.6745}{\sqrt{n(n-1)}}$.
			40	0.1080	0.0171
			41	1066	0167
2	0.6745	0.4769	42	1053	0163
3	4769	2754	43	1041	0159
4	3894	1947	44	1029	0155
5	0.3372	0.1508	45	0.1017	0.0152
6	3016	1231	46	1005	0148
7	2754	1041	47	0994	0145
8	2549	0901	48	0984	0142
9	2385	0795	49	0974	0139
10	0.2248	0.0711	50	0.0964	0.0136
11	2133	0643	51	0954	0134
12	2029	0587	52	0944	0131
13	1947	0540	53	0935	0128
14	1871	0500	54	0926	0126
15	0.1803	0.0465	55	0.0918	0.0124
16	1742	0435	56	0909	0122
17	1686	0409	57	0901	0119
18	1636	0386	58	0893	0117
19	1590	0365	59	0886	0115
20	0.1547	0.0346	60	0.0878	0.0113
21	1508	0329	61	0871	0111
22	1472	0314	62	0864	0110
23	1438	0300	63	0857	0108
24	1406	0287	64	0850	0106
25	0.1377	0.0275	65	0.0843	0.0105
26	1349	0265	66	0837	0103
27	1323	0255	67	0830	0101
28	1298	0245	68	0824	0100
29	1275	0237	69	0818	0098
30	0.1252	0.0229	70	0.0812	0.0097
31	1231	0221	71	0806	0096
32	1211	0214	72	0800	0094
33	1192	0208	73	0795	0093
34	1174	0201	74	0789	0092
35	0.1157	0.0196	75	0.0784	0.0091
36	1140	0190	80	0759	0085
37	1124	0185	85	0736	0080
38	1109	0180	90	0713	0075
39	1094	0175	100	0678	0068

TABLE IV.

For Computing Probable Errors by Formulas (35) and (36).

n.	$\frac{0.8453}{\sqrt{n(n-1)}}$.	$\frac{0.8453}{n\sqrt{n-1}}$.	n.	$\frac{0.8453}{\sqrt{n(n-1)}}$.	$\frac{0.8453}{n\sqrt{n-1}}$.
			40	0.0214	0.0034
			41	0209	0033
2	0.5978	0.4227	42	0204	0031
3	3451	1993	43	0199	0030
4	2440	1220	44	0194	0029
5	0.1890	0.0845	45	0.0190	0.0028
6	1543	0630	46	0186	0027
7	1304	0493	47	0182	0027
8	1130	0399	48	0178	0026
9	0996	0332	49	0174	0025
10	0.0891	0.0282	50	0.0171	0.0024
11	0806	0243	51	0167	0023
12	0736	0212	52	0164	0023
13	0677	0188	53	0161	0022
14	0627	0167	54	0158	0022
15	0.0583	0.0151	55	0.0155	0.0021
16	0546	0136	56	0152	0020
17	0513	0124	57	0150	0020
18	0483	0114	58	0147	0019
19	0457	0105	59	0145	0019
20	0.0434	0.0097	60	0.0142	0.0018
21	0412	0090	61	0140	0018
22	0393	0084	62	0137	0017
23	0376	0078	63	0135	0017
24	0360	0073	64	0133	0017
25	0.0345	0.0069	65	0.0131	0.0016
26	0332	0065	66	0129	0016
27	0319	0061	67	0127	0016
28	0307	0058	68	0125	0015
29	0297	0055	69	0123	0015
30	0.0287	0.0052	70	0.0122	0.0015
31	0277	0050	71	0120	0014
32	0268	0047	72	0118	0014
33	0260	0045	73	0117	0014
34	0252	0043	74	0115	0013
35	0.0245	0.0041	75	0.0113	0.0013
36	0238	0040	80	0106	0012
37	0232	0038	85	0100	0011
38	0225	0037	90	0095	0010
39	0220	0036	100	0085	0008

TABLE V.—Common Logarithms.

n	0	1	2	3	4	5	6	7	8	9	Diff.
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	42
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	38
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	35
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	32
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	30
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	28
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	27
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	25
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	24
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	22
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	21
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	20
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	19
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	18
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	18
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	17
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	17
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	16
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	15
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	15
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	14
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	14
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	13
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	13
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	13
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	12
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	12
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	12
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	11
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	11
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	11
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	11
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	10
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	10
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	10
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	10
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	9
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	9
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	9
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	9
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	9
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	8
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	8
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	8
n	0	1	2	3	4	5	6	7	8	9	Diff.

TABLE V.—Common Logarithms.

n	0	1	2	3	4	5	6	7	8	9	Diff.
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	8
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	7
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	
62	7924	7931	7938	7945	7952	7959	7866	7973	7980	7987	
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	7
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	6
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	5
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	
n	0	1	2	3	4	5	6	7	8	9	Diff.

TABLE VI.—Squares of Numbers.

n	0	1	2	3	4	5	6	7	8	9	Diff.
1.0	1.000	1.020	1.040	1.061	1.082	1.103	1.124	1.145	1.166	1.188	22
1.1	1.210	1.232	1.254	1.277	1.300	1.323	1.346	1.369	1.392	1.416	24
1.2	1.440	1.464	1.488	1.513	1.538	1.563	1.588	1.613	1.638	1.664	26
1.3	1.690	1.716	1.742	1.769	1.796	1.823	1.850	1.877	1.904	1.932	28
1.4	1.960	1.988	2.016	2.045	2.074	2.103	2.132	2.161	2.190	2.220	30
1.5	2.250	2.280	2.310	2.341	2.372	2.403	2.434	2.465	2.496	2.528	32
1.6	2.560	2.592	2.624	2.657	2.690	2.723	2.756	2.789	2.822	2.856	34
1.7	2.890	2.924	2.958	2.993	3.028	3.063	3.098	3.133	3.168	3.204	36
1.8	3.240	3.276	3.312	3.349	3.386	3.423	3.460	3.497	3.534	3.572	38
1.9	3.610	3.648	3.686	3.725	3.764	3.803	3.842	3.881	3.920	3.960	40
2.0	4.000	4.040	4.080	4.121	4.162	4.203	4.244	4.285	4.326	4.368	42
2.1	4.410	4.452	4.494	4.537	4.580	4.623	4.666	4.709	4.752	4.796	44
2.2	4.840	4.884	4.928	4.973	5.018	5.063	5.108	5.153	5.198	5.244	46
2.3	5.290	5.336	5.382	5.429	5.476	5.523	5.570	5.617	5.664	5.712	48
2.4	5.760	5.808	5.856	5.905	5.954	6.003	6.052	6.101	6.150	6.200	50
2.5	6.250	6.300	6.350	6.401	6.452	6.503	6.554	6.605	6.656	6.708	52
2.6	6.760	6.812	6.864	6.917	6.970	7.023	7.076	7.129	7.182	7.236	54
2.7	7.290	7.344	7.398	7.453	7.508	7.563	7.618	7.673	7.728	7.784	56
2.8	7.840	7.896	7.952	8.009	8.066	8.123	8.180	8.237	8.294	8.352	58
2.9	8.410	8.468	8.526	8.585	8.644	8.703	8.762	8.821	8.880	8.940	60
3.0	9.000	9.060	9.120	9.181	9.242	9.303	9.364	9.425	9.486	9.548	62
3.1	9.610	9.672	9.734	9.797	9.860	9.923	9.986	10.05	10.11	10.18	6
3.2	10.24	10.30	10.37	10.43	10.50	10.56	10.63	10.69	10.76	10.82	7
3.3	10.89	10.96	11.02	11.09	11.16	11.22	11.29	11.36	11.42	11.49	7
3.4	11.56	11.63	11.70	11.76	11.83	11.90	11.97	12.04	12.11	12.18	7
3.5	12.25	12.32	12.39	12.46	12.53	12.60	12.67	12.74	12.82	12.89	7
3.6	12.96	13.03	13.10	13.18	13.25	13.32	13.40	13.47	13.54	14.62	7
3.7	13.69	13.76	13.84	13.91	13.99	14.06	14.14	14.21	14.29	14.36	8
3.8	14.44	14.52	14.59	14.67	14.75	14.82	14.90	14.98	15.05	15.13	8
3.9	15.21	15.29	15.37	15.44	15.52	15.60	15.68	15.76	15.84	15.92	8
4.0	16.00	16.08	16.16	16.24	16.32	16.40	16.48	16.56	16.65	16.73	8
4.1	16.81	16.89	16.97	17.06	17.14	17.22	17.31	17.39	17.47	17.56	8
4.2	17.64	17.72	17.81	17.89	17.98	18.06	18.15	18.23	18.32	18.40	9
4.3	18.49	18.58	18.66	18.75	18.84	18.92	19.01	19.10	19.18	19.27	9
4.4	19.36	19.45	19.54	19.62	19.71	19.80	19.89	19.98	20.07	20.16	9
4.5	20.25	20.34	20.43	20.52	20.61	20.70	20.79	20.88	20.98	21.07	9
4.6	21.16	21.25	21.34	21.44	21.53	21.62	21.72	21.81	21.90	22.00	9
4.7	22.09	22.18	22.28	22.37	22.47	22.56	22.66	22.75	22.85	22.94	10
4.8	23.04	23.14	23.23	23.33	23.43	23.52	23.62	23.72	23.81	23.91	10
4.9	24.01	24.11	24.21	24.30	24.40	24.50	24.60	24.70	24.80	24.90	10
5.0	25.00	25.10	25.20	25.30	25.40	25.50	25.60	25.70	25.81	25.91	10
5.1	26.01	26.11	26.21	26.32	26.42	26.52	26.63	26.73	26.83	26.94	10
5.2	27.04	27.14	27.25	27.35	27.46	27.56	27.67	27.77	27.88	27.98	11
5.3	28.09	28.20	28.30	28.41	28.52	28.62	28.73	28.84	28.94	29.05	11
5.4	29.16	29.27	29.38	29.48	29.59	29.70	29.81	29.92	30.03	30.14	11
n	0	1	2	3	4	5	6	7	8	9	Diff.

TABLE VI.—Squares of Numbers.

n	0	1	2	3	4	5	6	7	8	9	Diff.
5.5	30.25	30.36	30.47	30.58	30.69	30.80	30.91	31.02	31.14	31.25	11
5.6	31.36	31.47	31.58	31.70	31.81	31.92	32.04	32.15	32.26	32.38	11
5.7	32.49	32.60	32.72	32.83	32.95	33.06	33.18	33.29	33.41	33.52	12
5.8	33.64	33.76	33.87	33.99	34.11	34.22	34.34	34.46	34.57	34.69	12
5.9	34.81	34.93	35.05	35.16	35.28	35.40	35.52	35.64	35.76	35.88	12
6.0	36.00	36.12	36.24	36.36	36.48	36.60	36.72	36.84	36 97	37.09	12
6.1	37.21	37.33	37.45	37.58	37.70	37.82	37.95	38.07	38.19	38.32	12
6.2	38.44	38.56	38.69	38.81	38.94	39.06	39.19	39.31	39.44	39.56	13
6.3	39.69	39.82	39.94	40.07	40.20	40 32	40.45	40.58	40.70	40.53	13
6.4	40.96	41.09	41.22	41.34	41.47	41.60	41.73	41.86	41.99	42.12	13
6.5	42.25	42.38	42.51	42.64	42.77	42.90	43.03	43.16	43.30	43.43	13
6.6	43.56	43.69	43.82	43.96	44.09	44.22	44.36	44.49	44.62	44.76	13
6 7	44.89	45.02	45.16	45.29	45.43	45.56	45 70	45.83	45.97	46.10	14
6.8	46.24	46.38	46.51	46.65	46.79	46.92	47.06	47.20	47.33	47.47	14
6.9	47.61	47.75	47.89	48.02	48.16	48.30	48.44	48.58	48.72	48.86	14
7.0	49.00	49.14	49.28	49.42	49.56	49.70	49.84	49.98	50.13	50.27	14
7.1	50.41	50.55	50.69	50.84	50.98	51.12	51.27	51.41	51.55	51.70	14
7.2	51.84	51.98	52.13	52.27	52.42	52.56	52.71	52.85	53.00	53.14	15
7.3	53.29	53.44	53.58	53.73	53.88	54.02	54.17	54.32	54.46	54.61	15
7.4	54.76	54.91	55.06	55.20	55.35	55.50	55.65	55.80	55.95	56.10	15
7.5	56.25	56.40	56.55	56.70	56.85	57.00	57.15	57.30	57.46	57.61	15
7.6	57.76	57.91	58.06	58.22	58.37	58 52	58.68	58.83	58.98	59.14	15
7.7	59.29	59.44	59.60	59.75	59.91	60.06	60.22	60.37	60.53	60.68	16
7.8	60.84	61.00	61.15	61.31	61.47	61.62	61.78	61.94	62.09	62.25	16
7.9	62.41	62.57	62.73	62.88	63.04	63.20	63.36	63.52	63.68	63 84	16
8.0	64.00	64.16	64.32	64.48	64.64	64.80	64 96	65.12	65.29	65.45	16
8.1	65.61	65.77	65.93	66.10	66.26	66.42	66.59	66.75	66.91	67.08	16
8.2	67.24	67.40	67.57	67.73	67.90	68.06	68.23	68.39	68.56	68.72	17
8.3	68.89	69.06	69.22	69.39	69.56	69.72	69.89	70.06	70.22	70.39	17
8.4	70.56	70.73	70.90	71.06	71.23	71.40	71.57	71.74	71.91	72.08	17
8.5	72.25	72.42	72.59	72.76	72.93	73.10	73.27	73.44	73.62	73.79	17
8.6	73.96	74.13	74.30	74.48	74.65	74.82	75.00	75.17	75.34	75.52	17
8.7	75.69	75.86	76.04	76.21	76.39	76.56	76.74	76.91	77.09	77.26	18
8.8	77.44	77.62	77.79	77.97	78.15	78.32	78.50	78.68	78.85	79.03	18
8.9	79.21	79.39	79.57	79.74	79.92	80.10	80.28	80.46	80.64	80 82	18
9.0	81.00	81.18	81.36	81.54	81.72	81.90	82.08	82.26	82.45	82.63	18
9.1	82.81	82.99	83.17	83.36	83.54	83.72	83.91	84.09	84.27	84.46	18
9.2	84.64	84.82	85.01	85.19	85.38	85.56	85.75	85.93	86.12	86.30	19
9.3	86.49	86.68	86.86	87.05	87.24	87.42	87.61	87.80	87.98	88.17	19
9.4	88.36	88.55	88.74	88.92	89.11	89.30	89.49	89.68	89.87	90.06	19
9.5	90.25	90.44	90.63	90.82	91.01	91.20	91.39	91.58	91.78	91.97	19
9.6	92.16	92.35	92.54	92.74	92.93	93.12	93.32	93.51	93.70	93.90	19
9.7	94.09	94.28	94.48	94.67	94.87	95.06	95.26	95.45	95.65	95.84	20
9.8	96.04	96.24	96.43	96.63	96.83	97 02	97.22	97.42	97.61	97.81	20
9.9	98.01	98.21	98.41	98.60	98.80	99.00	99.20	99.40	99.60	99.80	20
n	0	1	2	3	4	5	6	7	8	9	Diff.

TABLE VII. — For Applying Chauvenet's Criterion.

n.	t.	n.	t.	n.	t.
3	2.05	13	3.07	23	3.41
4	2.27	14	3.12	24	3.43
5	2.44	15	3.16	25	3.45
6	2.57	16	3.19	30	3.55
7	2.67	17	3.22	40	3.70
8	2.76	18	3.26	50	3.82
9	2.84	19	3.29	75	4.02
10	2.91	20	3.32	100	4.16
11	2.96	21	3.35	200	4.48
12	3.02	22	3.38	500	4.90

TABLE VIII. — Squares of Reciprocals.

n.	$\frac{1}{n^2}$	n.	$\frac{1}{n^2}$	n.	$\frac{1}{n^2}$
0.0	∞	2.5	0.1600	5.0	0.0400
0.1	100.000	2.6	0.1479	5.1	0.0384
0.2	25.000	2.7	0.1372	5.2	0.0370
0.3	11.111	2.8	0.1276	5.3	0.0356
0.4	6.250	2.9	0.1189	5.4	0.0343
0.5	4.000	3.0	0.1111	5.5	0.0331
0.6	2.778	3.1	0.1041	5.6	0.0319
0.7	2.041	3.2	0.0977	5.7	0.0308
0.8	1.562	3.3	0.0918	5.8	0.0297
0.9	1.235	3.4	0.0865	5.9	0.0287
1.0	1.000	3.5	0.0816	6.0	0.0278
1.1	0.8264	3.6	0.0772	6.1	0.0269
1.2	0.6944	3.7	0.0730	6.2	0.0260
1.3	0.5917	3.8	0.0693	6.3	0.0252
1.4	0.5102	3.9	0.0657	6.4	0.0244
1.5	0.4444	4.0	0.0625	6.5	0.0237
1.6	0.3906	4.1	0.0595	6.6	0.0230
1.7	0.3460	4.2	0.0567	6.7	0.0223
1.8	0.3086	4.3	0.0541	6.8	0.0216
1.9	0.2770	4.4	0.0517	6.9	0.0210
2.0	0.2500	4.5	0.0494	7.0	0.0204
2.1	0.2268	4.6	0.0473	7.5	0.0178
2.2	0.2066	4.7	0.0453	8.0	0.0156
2.3	0.1890	4.8	0.0434	8.5	0.0138
2.4	0.1736	4.9	0.0416	9.0	0.0123

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